ROMANIAN MATHEMATICAL MAGAZINE

UP.541 Find:

$$\Omega=\int_1^{\sqrt{3}}\frac{x-\tan^{-1}x}{(1+x^2)^2(\tan^{-1}x)^3}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let by $y=\tan^{-1}x\Rightarrow dy=\frac{1}{\cos^2 y}dy; x=\tan y$

For $x=1\Rightarrow y=\frac{\pi}{4}$. For $x=\sqrt{3}\Rightarrow y=\frac{\pi}{3}$.

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan y-y}{(1+\tan^2 y)^2\cdot y^3}\cdot\frac{1}{\cos^2 y}dy=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{\sin y}{\cos y}-y}{\frac{1}{\cos^4 y}\cdot y^3}\cdot\frac{1}{\cos^2 y}dy=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{(\sin y-y\cos y)\cos y}{y^3}dy=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin y\cos y-y\cos^2 y}{y^3}dy=-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sin y\cos y-y\cos^2 y)\cdot\left(\frac{-2}{y^3}\right)dy=$$

$$=-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sin y\cos y-y\cos^2 y)\cdot\left(\frac{1}{y^2}\right)'dy\overset{IBP}{=}-\frac{1}{2}(\sin y\cos y-y\cos^2 y)\cdot\frac{1}{y^2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}+$$

$$+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\cos^2 y-\sin^2 y-\cos^2 y+2y\sin y\cos y)\cdot\frac{1}{y^2}dy=$$

$$=-\frac{1}{2y^2}(\sin y\cos y-y\cos^2 y)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}+$$

$$+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(2y\sin y\cos y-\sin^2 y)\cdot\frac{1}{y^2}dy=-\frac{1}{2\cdot\frac{\pi^2}{9}}\left(\sin\frac{\pi}{3}\cos\frac{\pi}{3}-\frac{\pi}{3}\cos^2\frac{\pi}{3}\right)+$$

$$+\frac{1}{2\cdot\frac{\pi^2}{16}}\left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}-\frac{\pi}{4}\cos^2\frac{\pi}{4}\right)+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{\sin^2 y}{y}\right)'dy=$$

$$=-\frac{9}{2\pi^2}\left(\frac{\sqrt{3}}{4}-\frac{\pi}{12}\right)+\frac{8}{\pi^2}\left(\frac{1}{2}-\frac{\pi}{8}\right)+\frac{1}{2}\left(\frac{\sin^2\frac{\pi}{3}}{\frac{\pi}{3}}-\frac{\sin^2\frac{\pi}{4}}{\frac{\pi}{4}}\right)=$$

$$=\frac{1}{\pi^2}\left(-\frac{9\sqrt{3}}{8}+\frac{9\pi}{24}+4-\pi\right)+\frac{1}{\pi}\left(3\cdot\frac{3}{4}-4\cdot\frac{1}{2}\right)=\frac{1}{\pi^2}\left(4-\frac{9\sqrt{3}}{8}-\frac{5\pi}{8}\right)+\frac{1}{2\pi}\cdot\frac{1}{4}=$$

$$=\frac{1}{8\pi^2}(32-9\sqrt{3}-5\pi)+\frac{1}{8\pi}$$

Solution 2 by Marin Chirciu-Romania

Using the substitution $t=\arctan x \Rightarrow x=\tan t \Rightarrow dx=(1+\tan^2 t)dt \Rightarrow$

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t-t}{(1+\tan^2 t)^2t^3}(1+\tan^2 t)dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t-t}{(1+\tan^2 t)t^3}dt=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t}{(1+\tan^2 t)t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{t}{(1+\tan^2 t)t^3}dt=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{\sin t}{\cos t}}{\frac{1}{\cos^2 t}t^3}dt-\int_{\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{t}{\frac{1}{\cos^2 t}t^3}dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin t\cos t}{t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos^2 t}{t^2}dt=$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin 2t}{t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1+\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{-1}{2t^2}\right)'\sin 2\,dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2t^2}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\left[\left(\frac{-1}{2t^2}\right)\sin 2t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{-1}{2t^2}\right)\cdot 2\cos 2t\,dt\right]-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2t^2}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}=$$

$$=\frac{1}{2}\left[\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\sin\frac{2\pi}{3}+\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\sin\frac{2\pi}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t\,dt\right]+\frac{1}{2t}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\left[\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\sin\frac{2\pi}{3}+\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\sin\frac{2\pi}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t\,dt\right]+\frac{1}{2\cdot\frac{\pi}{3}}-\frac{1}{2\cdot\frac{\pi}{4}}$$

$$-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt$$

$$=\frac{1}{2}\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\cdot 1+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t\,dt+\frac{1}{\frac{2\pi}{3}}-\frac{1}{\frac{2\pi}{4}}-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{t^2}dt=$$

$$=\frac{-\sqrt{3}}{8\left(\frac{\pi}{3}\right)^2}+\frac{1}{4\left(\frac{\pi}{4}\right)^2}+\frac{3}{2\pi}-\frac{4}{2\pi}=\frac{-\sqrt{3}}{8\frac{\pi^2}{9}}+\frac{1}{4\frac{\pi^2}{16}}-\frac{1}{2\pi}=\frac{-9\sqrt{3}}{8\pi^2}+\frac{4}{\pi^2}-\frac{1}{2\pi}$$

We deduce that $\Omega=\int_1^{\sqrt{3}}\frac{x-\arctan x}{(1+x^2)(\arctan x)^3}dx=\frac{4\pi}{\pi^2}-\frac{9\sqrt{3}}{8\pi^2}-\frac{1}{2\pi}$

Remark: The problem can be developed.

If $k\in\mathbb{N}$ find:

$$\Omega=\int_1^{\sqrt{3}}\frac{kx-\arctan x}{(1+x^2)^2(\arctan x)^{2k+1}}dx$$

Marin Chirciu

Solution

For $k=0$ we have $\Omega=\int_1^{\sqrt{3}}\frac{-\arctan x}{(1+x^2)(\arctan x)}dx=-\int_1^{\sqrt{3}}\frac{1}{(1+x^2)}dx$

Integrating by parts we have

$$\int\frac{1}{x^2+1}dx=\int x'\frac{1}{x^2+1}dx=x\cdot\frac{1}{x^2+1}-\int x\left(\frac{1}{x^2+1}\right)'dx=$$

$$=\frac{x}{x^2+1}-\int x\cdot\frac{-2x}{(x^2+1)^2}dx=$$

$$=\frac{x}{x^2+1}+2\frac{x^2}{(x^2+1)^2}dx=\frac{x}{x^2+1}+2\int\frac{x^2+1-1}{(x^2+1)^2}dx=\frac{x}{x^2+1}+2\int\frac{1}{x^2+1}-$$

$$-2\int\frac{1}{(x^2+1)^2}dx$$

It follows $\int\frac{1}{x^2+1}dx=\frac{x}{x^2+1}+2\int\frac{1}{x^2+1}-2\int\frac{1}{(x^2+1)}dx\Leftrightarrow 2\int\frac{1}{(x^2+1)^2}dx=\frac{x}{x^2+1}+\int\frac{1}{x^2+1}$

wherefrom $\int\frac{1}{(x^2+1)^2}dx=\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\arctan x$

We obtain

$$\int_1^{\sqrt{3}}\frac{1}{(1+x^2)^2}dx=\left(\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\arctan x\right)\Big|_1^{\sqrt{3}}=$$

$$=\frac{1}{2}\cdot\frac{\sqrt{3}}{3+1}+\frac{1}{2}\arctan\sqrt{3}-\frac{1}{2}\cdot\frac{1}{1+1}-\frac{1}{2}\arctan 1=$$

$$=\frac{\sqrt{3}}{8}+\frac{1}{2}\cdot\frac{\pi}{3}-\frac{1}{4}-\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\sqrt{3}}{8}-\frac{1}{4}+\frac{1}{2}\cdot\frac{\pi}{12}=\frac{\sqrt{3}}{8}-\frac{1}{4}+\frac{\pi}{24}$$

$$\Omega=-\int_1^{\sqrt{3}}\frac{1}{(1+x^2)^2}dx=-\left(\frac{\sqrt{3}}{8}-\frac{1}{4}+\frac{\pi}{24}\right)=\frac{1}{4}-\frac{\sqrt{3}}{8}-\frac{\pi}{24}.$$

Next, let be $k \geq 1$.

Using the substitution $t = \arctan x \Rightarrow x = \tan t \Rightarrow dx = (1+\tan^2 t)dt \Rightarrow$

$$\Omega = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{k\tan t - 1}{(1+\tan^2 t)^2}(1+\tan^2 t)dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{k\tan t - t}{(1+\tan^2 t)t^{2k+1}}dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{k\tan t}{(1+\tan^2 t)t^{2k+1}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{t}{(1+\tan^2 t)t^{2k+1}}dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{k\frac{\sin t}{\cos t}}{\frac{1}{\cos^2 t}t^{2k+1}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{t}{\frac{1}{\cos^2 t}t^{2k+1}}dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{k\sin t\cos t}{t^{2k+1}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2 t}{t^{2k+1}} = \frac{k}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2t}{t^{2k+1}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\cos 2t}{2t^{2k}} =$$

$$= \frac{k}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{-1}{2kt^{2k}}\right)' \sin 2t\,dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2t^{2k}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t}{2t^{2k}} =$$

$$= \frac{k}{2}\left[\left(-\frac{1}{2kt^{2k}}\right)\sin 2t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{-1}{2kt^{2k}}\right)\cdot 2\cos 2t\,dt\right] -$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2t^{2k}}dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t}{2t^{2k}}dt = \frac{k}{2}\left[\left(-\frac{1}{2k\left(\frac{\pi}{3}\right)^{2k}}\right)\sin\frac{2\pi}{3} + \left(\frac{1}{2k\left(\frac{\pi}{4}\right)^{2k}}\right)\sin\frac{2\pi}{4} + + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{2kt^{n-1}}\right)\cdot 2\cos 2t\,dt\right] +$$

$$+ \frac{1}{2(2k-1)t^{2k-1}}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t}{2t^{2k}}dt = \frac{k}{2}\left[\left(\frac{-1}{2k\left(\frac{\pi}{3}\right)^{2k}}\right)\cdot\frac{\sqrt{3}}{2} + \left(\frac{1}{2\left(\frac{\pi}{4}\right)^{2}}\right)\cdot 1 + + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{t^2}\cos 2t\,dt\right] +$$

$$+ \frac{1}{2(2k-1)\cdot\left(\frac{\pi}{3}\right)^{2k-1}} - \frac{1}{2\cdot\left(\frac{\pi}{4}\right)^{2k-1}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t}{2t^2}dt =$$

$$= \frac{-\frac{k}{2}}{2k\left(\frac{\pi}{3}\right)^{2k}}\cdot\frac{\sqrt{3}}{2} + \frac{\frac{k}{2}}{2k\left(\frac{\pi}{4}\right)^{2k}}\cdot 1 + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2t^2}\cos 2t\,dt +$$

$$+\frac{1}{2(2k-1)\left(\frac{\pi}{3}\right)^{2k-1}}-\frac{1}{2(2k-1)\left(\frac{\pi}{4}\right)^{2k-1}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{-\sqrt{3}}{8\left(\frac{\pi}{3}\right)^{2k}}+\frac{1}{4\left(\frac{\pi}{4}\right)^{2k}}+\frac{1}{2(2k-1)\left(\frac{\pi}{3}\right)^{2k-1}}-\frac{1}{2(2k-1)\left(\frac{\pi}{4}\right)^{2k-1}}=$$

$$=\frac{-\sqrt{3}\cdot 3^{2k}}{8\pi^{2k}}+\frac{4^{2k-1}}{\pi^{2k}}+\frac{3^{2k-1}-4^{2k-1}}{2(2k-1)\pi^{2k-1}}=$$

We deduce that $\Omega=\int_1^{\sqrt{3}}\frac{kx-\arctan x}{(1+x^2)(\arctan x)^{2k+1}}dx=\frac{4^{2k-1}}{\pi^{2k}}-\frac{\sqrt{3}\cdot 3^{2k}}{8\pi^{2k}}+\frac{3^{2k-1}-4^{2k-1}}{2(2k-1)\pi^{2k-1}}$

Note: For $k=1$ we obtain Problem UP.541 from RMM Nr.37 – Summer 2025.