UP.542 Find:

$$\Omega=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{49}{49+(7n+1)(7n+8)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{7\big((7n+8)-(7n+1)\big)}{49+(7n+1)(7n+8)}\right)=$$

$$=\sum_{n=1}^{\infty}\left(\frac{(7n+1)(7n+8)\cdot 7\cdot\left(\frac{1}{7n+1}-\frac{1}{7n+8}\right)}{(7n+1)(7n+8)\left(\frac{49}{(7n+1)(7n+8)}+1\right)}\right)=$$

$$=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\frac{7}{7n+1}-\frac{7}{7n+8}}{1+\frac{7}{7n+1}\cdot\frac{7}{7n+8}}\right)=\sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{7}{7n+1}\right)-\tan^{-1}\left(\frac{7}{7n+8}\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{7}{7k+1}\right)-\tan^{-1}\left(\frac{7}{7k+8}\right)\right)=\lim_{n\to\infty}\left(\tan^{-1}\left(\frac{7}{8}\right)-\tan^{-1}\left(\frac{7}{7n+8}\right)\right)=$$

$$=\tan^{-1}\left(\frac{7}{8}\right)-\tan^{-1}0=\tan^{-1}\left(\frac{7}{8}\right)$$

Solution 2 by Marin Chirciu-Romania

Using $\arctan\frac{1}{x}-\arctan\frac{1}{x+1}=\arctan\frac{1}{x^2+x+1}, x>0$, for $x=n+\frac{1}{7}$ we obtain:

$$\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+\frac{1}{7}+1}=\arctan\frac{1}{\left(n+\frac{1}{7}\right)^2+n+\frac{1}{7}+1}$$

$$=\frac{49}{49+(7n+1)(7n+8)}$$

It follows:

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{49}{49+(7n+1)(7n+8)}=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+\frac{1}{7}+1}\right)=$$

$$=\arctan\frac{1}{1+\frac{1}{7}}-\arctan\frac{1}{2+\frac{1}{7}}+\arctan\frac{1}{2+\frac{1}{7}}-\arctan\frac{1}{3+\frac{1}{7}}+\cdots+$$

$$+\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+1+\frac{1}{7}}+\cdots=$$

$$=\lim_{n\to\infty}\left(\arctan\frac{1}{1+\frac{1}{7}}-\arctan\frac{1}{n+1+\frac{1}{7}}\right)=\arctan\frac{7}{8}$$

Remark.

The problem can be developed.

Let $\lambda>0$. Find:

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}$$

Marin Chirciu

Solution

Using $\arctan\frac{1}{x}-\arctan\frac{1}{x+1}=\arctan\frac{1}{x^2+x+1}, x>0$, for $x=n+\frac{1}{\lambda}$ we obtain:

$$\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+\frac{1}{\lambda}+1}=\arctan\frac{1}{\left(n+\frac{1}{\lambda}\right)^2+n+\frac{1}{\lambda}+1}=$$

$$=\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}$$

It follows:

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+\frac{1}{\lambda}+1}\right)$$

$$=$$

$$=\arctan\frac{1}{1+\frac{1}{\lambda}}-\arctan\frac{1}{2+\frac{1}{\lambda}}+\arctan\frac{1}{2+\frac{1}{\lambda}}-\arctan\frac{1}{3+\frac{1}{\lambda}}+\cdots+$$

$$+\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+1+\frac{1}{\lambda}}+\cdots=$$

$$=\lim_{n\to\infty}\left(\arctan\frac{1}{1+\frac{1}{\lambda}}-\arctan\frac{1}{n+1+\frac{1}{\lambda}}\right)=\arctan\frac{\lambda}{\lambda+1}$$

Note: For $\lambda=7$ we obtain Problem SP.542 from RMM Nr. 37 – Summer 2025.