ROMANIAN MATHEMATICAL MAGAZINE

UP.545 Find:

$$\Omega=\int_0^{\frac{1}{2}}\frac{x^5-3x^3}{3x^6-x^4-3x^2+1}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x\in\left(0,\frac{1}{2}\right)\Rightarrow(\exists)y\in\left(0,\arctan\frac{1}{2}\right);x=\tan y$$

$$dx=\frac{1}{\cos^2 y}dy;dx=(1+\tan^2 y)dy$$

$$x=0\Rightarrow y=0;x=\frac{1}{2}\Rightarrow y=\arctan\frac{1}{2}$$

$$\Omega=\int_0^{\arctan\frac{1}{2}}\frac{(\tan^5 y-3\tan^3 y)(1+\tan^2 y)}{3\tan^6 y-\tan^4 y-3\tan^2 y+1}dy$$

$$\Omega=-\int_0^{\arctan\frac{1}{2}}\frac{\tan^3 y\,(3-\tan^2 y)(1+\tan^2 y)}{3\tan^2 y\,(\tan^4 y-1)-(\tan^4 y-1)}dy$$

$$\Omega=-\int_0^{\arctan\frac{1}{2}}\frac{\tan^3 y\,(3-\tan^2 y)(1+\tan^2 y)}{(\tan^2 y-1)(\tan^2 y+1)(3\tan^2 y-1)}dy$$

$$\Omega=\int_0^{\arctan\frac{1}{2}}\frac{\tan y\,(3-\tan^2 y)}{3\tan^2 y-1}\cdot\frac{\tan^2 y}{\tan^2 y-1}dy$$

$$\Omega=\frac{1}{2}\int_0^{\arctan\frac{1}{2}}\tan(3y)\cdot\frac{2\tan y}{\tan^2 y-1}\cdot\tan y\,dy$$

$$\Omega=\frac{1}{2}\int_0^{\arctan\frac{1}{2}}\tan(3y)\cdot\tan(2y)\cdot\tan y\,dy$$

$$\Omega=\frac{1}{2}\int_0^{\arctan\frac{1}{2}}(\tan 3y-\tan 2y-\tan y)\,dy$$

$$\Omega=\frac{1}{2}\int_0^{\arctan\frac{1}{2}}\tan 3y\,dy-\frac{1}{2}\int_0^{\arctan\frac{1}{2}}\tan 2y\,dy-\frac{1}{2}\int_0^{\arctan\frac{1}{2}}\tan y\,dy$$

$$\Omega=\frac{1}{2}\ln\left(\cos\left(\arctan\frac{1}{2}\right)\right)+\frac{1}{4}\ln\left(\cos\left(2\arctan\frac{1}{2}\right)\right)-\frac{1}{6}\ln\left(\cos\left(3\arctan\frac{1}{2}\right)\right)$$

Solution 2 by Marin Chirciu-Romania

With the substitution $x^2 = t$ we obtain:

$$\Omega = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{t^2 - 3t}{3t^3 - t^2 - 3t + 1} dt = \frac{1}{2}\int_1^{\frac{1}{4}} \frac{t^2 - 3t}{(t-1)(t+1)(3t-1)} dt =$$

$$= \frac{1}{2}\int_1^{\frac{1}{4}} \left(\frac{-\frac{1}{2}}{t-1} + \frac{-1}{3t-1} + \frac{-1}{t+1} \right) dt =$$

$$= \frac{1}{2}\left(-\frac{1}{2} ln|t-1| - \frac{1}{3} ln|3t-1| - ln|t+1|\right)\Big|_0^{\frac{1}{4}} =$$

$$= \frac{-1}{2}\left(\frac{1}{2}\ln|t-1| + \frac{1}{3}\ln|3t-1| + \ln|t+1|\right)\Big|_0^{\frac{1}{4}} =$$

$$= -\frac{1}{2}\left(\frac{1}{2}\ln\left|\frac{1}{4} - 1\right| + \frac{1}{3}\ln\left|\frac{3}{4} - 1\right| + \ln\left|\frac{1}{4} + 1\right|\right) = -\frac{1}{2}\left(\frac{1}{2}\ln\frac{3}{4} + \frac{1}{3}\ln\frac{1}{4} + \ln\frac{5}{4}\right) =$$

$$= -\frac{1}{2}\left(\frac{1}{2}\ln 3 - \ln 2 - \frac{2}{3}\ln 2 + \ln 5 + \ln 2 - 2\ln 2\right) = -\frac{1}{2}\left(\frac{1}{2}\ln 3 - \frac{11}{3}\ln 2 + \ln 5\right) =$$

$$= \frac{11}{6}\ln 2 - \frac{1}{4}\ln 3 - \frac{1}{2}\ln 5$$

I broke it down into simple fractions:

$$\frac{t^2-3t}{(t-1)(3t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{3t-1} + \frac{C}{t+1}, \text{ with } A = -\frac{1}{2}, B = -1, C = -1$$