ROMANIAN MATHEMATICAL MAGAZINE

UP.546. Prove without any software:

$$\int_0^2 \sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\,dx+\int_0^2 \sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\,dx>\frac{5}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:[0,1]\to[0,2]; f(x)=x^3+x$

f continuous; f – increasing because $f'(x)=3x^2+1>0$

f bijectif; $f^{-1}(x)=\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}$

$f(0)=0; f(1)=2$

By Young's inequality – integral form:

$$\int_0^1 (x^3+x)\,dx+\int_0^2\left(\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\right)dx\geq 1\cdot 2$$

$$\frac{1}{4}+\frac{1}{2}+\int_0^2\left(\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\right)dx+\int_0^2\left(\sqrt{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\right)dx>2$$

$$\int_0^2 \sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\,dx+\int_0^2 \sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\,dx>\frac{5}{4}$$

Solution 2 by Marin Chirciu-Romania

We use Cardano's formulas for the cubic equation $x^3+px+q=0$.

The discriminator is $\Delta=-(4p^3+27q^2)$

If $\Delta=-(4p^3+27q^2)<0$, the equation has a single real root.

$$x_1=\sqrt[3]{\frac{-q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{\frac{-q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}\text{ and } x_{2,3}\in\mathbb{C}-\mathbb{R}.$$

For $p=1,q=-1$ the third-grade equation $x^3+x-1=0$ has $\Delta=-(4+27)<0\Rightarrow$

$$x_1=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{27}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{1}{27}}}\text{ and } x_{2,3}\in\mathbb{C}-\mathbb{R}$$

We have:

$$f(x)=\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\geq$$

$$\geq f(1)=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{27}+\frac{1}{4}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{27}+\frac{1}{4}}}=x_1$$

We prove that $x_1>\frac{5}{8}$

We consider the function $g:[0,6]\to\mathbb{R}, g(x)=x^3+x-1$; we have

$$g'(x)=3x^2+1>0\Rightarrow g\uparrow$$

Because $g\left(\frac{5}{8}\right)=\left(\frac{5}{8}\right)^3+\frac{5}{8}-1=\frac{-323}{512}<0$ and $g(1)=1>0\Rightarrow\frac{5}{8}<x_1<1.$

We obtain:

$$\int_0^2\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx+\int_0^2\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx=$$

$$=\int_0^2\left(\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\right)dx>$$

$$>\int_0^2 x_1dx>\int_0^2\frac{5}{8}dx=\frac{5}{8}x\Big|_0^2=\frac{5}{8}\cdot 2=\frac{5}{4}$$