UP.547 Find without any software:

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}=\left(4-\frac{6}{x^3}-\frac{9}{x^4}\right)e^{2x+\frac{3}{x}}=\left(\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)-\frac{6}{x^3}\right)e^{2x+\frac{3}{x}}=$$

$$=-\frac{6}{x^3}e^{2x+\frac{3}{x}}+\left(2+\frac{3}{x^2}\right)\left(2-\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}=\left(2+\frac{3}{x^2}\right)'e^{2x+\frac{3}{x}}+\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)'=$$

$$=\left(\left(2+\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}\right)'$$

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx=\int_1^2\left(\left(2+\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}\right)'dx=$$

$$=\left(2+\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}\Big|_1^2=\frac{11}{4}\sqrt{e^{11}}-5e^5$$

Solution 2 by Marin Chirciu-Romania

We have $\frac{4x^4-6x-9}{x^4}=4-\frac{6}{x^3}-\frac{9}{x^4}=\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)-\frac{6}{x^3}$.

We obtain:

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx=\int_1^2\left(\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)-\frac{6}{x^3}\right)e^{2x+\frac{3}{x}}dx=$$

$$=\int_1^2\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=\int_1^2\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)'dx-$$

$$-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx\overset{Parts}{=}\int_1^2\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)'dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=$$

$$=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2-\int_1^2\left(2+\frac{3}{x^2}\right)'e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=$$

$$=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2+\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2=$$

$$=\left(2+\frac{3}{4}\right)\left(e^{4+\frac{3}{2}}\right)-\left(2+\frac{3}{1}\right)\left(e^{2+\frac{3}{1}}\right)=\frac{11}{4}e^{\frac{11}{2}}-5e^{5}$$

Remark.

The problem can be developed.

Let $a,b\in\mathbb{R}^{*}$. Find:

$$\Omega=\int_1^2\frac{a^2x^4-2bx-b^2}{x^4}e^{ax+\frac{b}{x}}dx$$

Marin Chirciu

Solution

We have $\frac{a^2x^4-2bx-b^2}{x^4}=a^2-\frac{2b}{x^3}-\frac{b^2}{x^4}=\left(a-\frac{b}{x^2}\right)\left(a+\frac{b}{x^2}\right)-\frac{2b}{x^3}$**.**

We obtain:

$$\Omega=\int_1^2\frac{a^2x^4-2bx-b^2}{x^4}e^{ax+\frac{b}{x}}dx\int_1^2\left(\left(a-\frac{b}{x^2}\right)\left(a+\frac{b}{x^2}\right)-\frac{2b}{x^3}\right)e^{ax+\frac{b}{x}}dx=$$

$$=\int_1^2\left(a-\frac{b}{x^2}\right)\left(a+\frac{b}{x^2}\right)e^{ax+\frac{b}{x}}dx-\int_1^2\frac{2b}{x^3}e^{ax+\frac{b}{x}}dx=\int_1^2\left(a+\frac{b}{x^2}\right)\left(e^{ax+\frac{b}{x}}\right)'dx-$$

$$-\int_1^2\frac{2b}{x^3}e^{ax+\frac{3}{x}}dx\overset{Parts}{=}\int_1^2\left(a+\frac{b}{x^2}\right)\left(e^{ax+\frac{b}{x}}\right)'dx-\int_1^2\frac{2b}{x^3}e^{ax+\frac{b}{x}}dx=$$

$$=\left(a+\frac{b}{x^2}\right)\left(e^{ax+\frac{b}{x}}\right)\Big|_1^2-\int_1^2\left(a+\frac{b}{x^2}\right)'e^{ax+\frac{b}{x}}dx-\int_1^2\frac{2b}{x^3}e^{ax+\frac{b}{x}}dx=$$

$$=\left(a+\frac{b}{x^2}\right)\left(e^{ax+\frac{b}{x}}\right)\Big|_1^2+\int_1^2\frac{2b}{x^3}e^{ax+\frac{3}{x}}dx-\int_1^2\frac{2b}{x^3}e^{ax+\frac{b}{x}}dx=\left(a+\frac{b}{x^2}\right)\left(e^{ax+\frac{b}{x}}\right)\Big|_1^2=$$

$$=\left(a+\frac{b}{4}\right)\left(e^{2a+\frac{b}{2}}\right)-\left(a+\frac{b}{1}\right)\left(e^{a+\frac{b}{1}}\right)=\frac{4a+b}{4}e^{\frac{4a+b}{2}}-(a+b)e^{a+b}$$

Note: For $a=2,b=3$ we obtain the problem UP.547 from RMM – 37 – Summer 2025