ROMANIAN MATHEMATICAL MAGAZINE

UP.548 Find:

$$\Omega=\int_1^e \frac{1-\ln x}{x^2+\ln^2 x}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega=\int_1^e \frac{1-\ln x}{x^2+\ln^2 x}dx=\int_1^e \frac{1-\ln x+1-\ln x}{(x-\ln x)^2+(x+\ln x)^2}dx=$$

$$=\int_1^e \frac{x-\ln x+1-\frac{\ln x}{x}-x+1-\ln x+\frac{\ln x}{x}}{(x-\ln x)^2\cdot\left(1+\left(\frac{x+\ln x}{x-\ln x}\right)^2\right)}dx=$$

$$=\int_1^e \frac{\left(1+\frac{1}{x}\right)(x-\ln x)-(x+\ln x)\left(1-\frac{1}{x}\right)}{(x-\ln x)^2}\cdot\frac{1}{1+\left(\frac{x+\ln x}{x-\ln x}\right)^2}dx=$$

$$=\int_1^e \frac{\left(\frac{x+\ln x}{x-\ln x}\right)'}{1+\left(\frac{x+\ln x}{x-\ln x}\right)^2}dx=\arctan\left(\frac{x+\ln x}{x-\ln x}\right)\Big|_1^e=$$

$$=\arctan\left(\frac{e+1}{e-1}\right)-\arctan\left(\frac{1+\ln 1}{1-\ln 1}\right)=\arctan\left(\frac{e+1}{e-1}\right)-\frac{\pi}{4}$$

Solution 2 by Marin Chirciu-Romania

$$\Omega=\int_1^e \frac{1-\ln x}{x^2+\ln^2 x}dx=\int_1^e \frac{\frac{1-\ln x}{\ln^2 x}}{\left(\frac{x}{\ln x}\right)^2+1}dx=\int_1^e \frac{-\left(\frac{x}{\ln x}\right)'}{\left(\frac{x}{\ln x}\right)^2+1}dx=-\arctan\frac{x}{\ln x}\Big|_1^e=$$

$$=-\left(\arctan\frac{e}{\ln e}-\arctan\frac{1}{\ln 1}\right)=-\left(\arctan e-\frac{\pi}{2}\right)=\frac{\pi}{2}-\arctan e.$$

Remark.

In the same way.

Find:

$$\Omega=\int_1^e \frac{1-\ln x}{x^2-\ln^2 x}dx$$

Marin Chirciu

Solution

$$\Omega=\int_1^e\frac{1-\ln x}{x^2-\ln^2 x}dx=\int_1^e\frac{\frac{1-\ln x}{\ln^2 x}}{\left(\frac{x}{\ln x}\right)^2-1}dx=\int_1^e\frac{-\left(\frac{x}{\ln x}\right)'}{\left(\frac{x}{\ln x}\right)^2-1}dx=-\frac{1}{2}\ln\left|\frac{\frac{x}{\ln x}-1}{\frac{x}{\ln x}+1}\right|\Bigg|_1^e=$$

$$=-\frac{1}{2}\ln\left|\frac{x-\ln x}{x+\ln x}\right|\Bigg|_1^e=-\frac{1}{2}\ln\left|\frac{x-\ln x}{x+\ln x}\right|\Bigg|_1^e=-\frac{1}{2}\left(\ln\frac{e-\ln e}{e+\ln e}-\ln\frac{1-\ln 1}{1+\ln 1}\right)=$$

$$=-\frac{1}{2}\left(\ln\frac{e-1}{e+1}-\ln 1\right)=-\frac{1}{2}\ln\frac{e-1}{e+1}=\frac{1}{2}\ln\frac{e+1}{e-1}$$