ROMANIAN MATHEMATICAL MAGAZINE

UP.549 If $0 < a \le b$ then find:

$$\Omega(a,b)=\int_a^b \frac{\ln x}{x^2+(a+b)x+ab}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $y=\frac{ab}{x}\Rightarrow x=\frac{ab}{y}\Rightarrow dx=-\frac{ab}{y^2}dy$

If $x=a$ then $y=b$.

If $x=b$ then $y=a$.

$$\Omega(a,b)=\int_a^b \frac{\ln\left(\frac{ab}{y}\right)}{\frac{a^2b^2}{y^2}+\frac{(a+b)ab}{y}+ab}\cdot\frac{-ab}{y^2}dy$$

$$\Omega(a,b)=-\int_b^a \frac{\ln(ab)-\ln y}{\left(\frac{ab}{y^2}+\frac{a+b}{y}+1\right)y^2}dy$$

$$\Omega(a,b)=\int_a^b \frac{\ln(ab)-\ln y}{ab+(a+b)y+y^2}dy$$

$$\Omega(a,b)=\int_a^b \frac{\ln(ab)}{y^2+(a+b)y+ab}dy-\int_a^b \frac{\ln y}{y^2+(a+b)y+ab}dy$$

$$\Omega(a,b)=\ln(ab)\int_a^b \frac{dy}{\left(y+\frac{a+b}{2}\right)^2-\frac{(a+b)^2-4ab}{4}}-\Omega(a,b)$$

$$2\Omega(a,b)=\ln(ab)\cdot\int_a^b \frac{dy}{\left(y+\frac{a+b}{2}\right)^2-\frac{(a-b)^2}{4}}$$

$$\Omega(a,b)=\frac{1}{2}\ln(ab)\cdot\frac{1}{2\cdot\frac{b-a}{2}}\ln\left|\frac{y+\frac{a+b}{2}-\frac{a-b}{2}}{y+\frac{a+b}{2}+\frac{a-b}{2}}\right|\Bigg|_a^b$$

$$\Omega(a,b)=\frac{\ln(ab)}{2(b-a)}\left(\ln\left|\frac{a+b}{2b}\right|-\ln\left|\frac{2a}{a+b}\right|\right)$$

$$\Omega(a,b)=\frac{\ln(ab)}{2(b-a)}\ln\frac{(a+b)^2}{4ab}$$

Solution 2 by Marin Chirciu-Romania

Using the substitution $x=\frac{ab}{t}\Rightarrow dx=\frac{-ab}{t^2}dt\Rightarrow$

$$\Rightarrow\Omega(a,b)=\int_a^b\frac{\ln x}{x^2+(a+b)x+ab}dx=\int_a^b\frac{\ln\frac{ab}{t}}{\left(\frac{ab}{t}\right)^2+(a+b)\frac{ab}{t}+ab}\left(-\frac{ab}{t^2}dt\right)=$$

$$=\int_a^b\frac{\ln(ab)-\ln t}{t^2+(a+b)t+ab}dt=\int_a^b\frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\int_a^b\frac{\ln t}{t^2+(a+b)t+ab}dt=$$

$$=\int_a^b\frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\Omega(a,b).$$

It follows that:

$$\Omega(a,b)=\int_a^b\frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\Omega(a,b)\Leftrightarrow$$

$$\Leftrightarrow 2\Omega(a,b)=\int_a^b\frac{\ln(ab)}{t^2+(a+b)t+ab}dt\Leftrightarrow\Omega(a,b)=\frac{\ln(ab)}{2}\int_a^b\frac{1}{t^2+(a+b)t+ab}dt$$

$$\int_a^b\frac{1}{t^2+(a+b)t+ab}dt=\int_a^b\frac{1}{(t+a)(t+b)}dt=\frac{1}{b-a}\int_a^b\left(\frac{1}{t+a}-\frac{1}{t+b}\right)dt=$$

$$=\frac{1}{b-a}\int_a^b\left(\frac{1}{t+a}-\frac{1}{t+b}\right)dt=\frac{1}{b-a}(\ln(t+a)-\ln(t+b))=$$

$=\frac{1}{b-a}\ln\left(\frac{t+a}{t+b}\right)\Big|_a^b=\frac{1}{b-a}\left(\ln\left(\frac{b+a}{b+b}\right)-\ln\left(\frac{a+a}{a+b}\right)\right)=\frac{1}{b-a}\ln\frac{(a+b)^2}{4ab}$**, for** $a<b$

For $a=b$ **we have** $\int_a^b\frac{1}{t^2+(a+b)t+ab}dt=0.$

We deduce that:

$$\Omega(a,b)=\frac{\ln(ab)}{2}\int_a^b\frac{1}{t^2+(a+b)t+ab}dt=\frac{\ln(ab)}{2}\cdot\frac{1}{b-a}\ln\frac{(a+b)^2}{4ab}=$$

$$=\frac{\ln(ab)}{2(b-a)}\ln\frac{(a+b)^2}{4ab}$$

for $a<b$ **and** $\Omega(a,b)=0$ **for** $a=b$.

Remark: In the same way.

If $0 < a \leq b$ then find:

$$\Omega(a,b) = \int_a^b \frac{\ln x}{(x+a)^2} dx$$

Marin Chirciu

Solution

Using the substitution $x = \frac{a^2}{t} \Rightarrow dx = \frac{-a^2}{t^2} dt \Rightarrow$

$$\Omega(a,b) = \int_a^b \frac{\ln x}{(x+a)^2} dx = \int_a^b \frac{\ln \frac{a^2}{t}}{\left(\frac{a^2}{t} + a\right)^2} \left(-\frac{a^2}{t^2} dt\right) = \int_a^b \frac{\ln(a^2) - \ln t}{(t+a)^2} dt =$$

$$= \int_a^b \frac{\ln(a^2)}{(t+a)^2} dt - \int_a^b \frac{\ln t}{(t+a)^2} dt = \int_a^b \frac{2\ln a}{(t+a)^2} dt - \Omega(a,b)$$

It follows that: $\Omega(a,b) = \int_a^b \frac{2\ln a}{(t+a)^2} - \Omega(a,b) \Leftrightarrow 2\Omega(a,b) = 2\ln a \int_a^b \frac{1}{(t+a)^2} dt \Leftrightarrow$

$$\Leftrightarrow \Omega(a,b) = \ln a \cdot \int_a^b \frac{1}{(t+a)^2} dt$$

$$\int_a^b \frac{1}{(t+a)^2} dt = \frac{-1}{t+a}\bigg|_a^b = \frac{-1}{b+a} + \frac{1}{2a} = \frac{-2a+b+a}{2a(a+b)} = \frac{b-a}{2a(a+b)}$$

We deduce that: $\Omega(a,b) = \ln a \cdot \frac{b-a}{2a(a+b)} = \frac{(b-a)\ln a}{2a(a+b)}$