UP.550 If $f:[0,1]\to\mathbb{R}$; f continuous and

$$\int_0^1 xf(x)\,dx=a;\ \int_0^1 f(x)\,dx=b;\ a,b\in\mathbb{R}$$

then:

$$\int_0^1 f^2(x)\,dx\ge 3(a-b)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\int_0^1 f^2(x)\,dx\cdot\int_0^1 (x-1)^2\,dx\overset{CBS}{\ge}\left(\int_0^1 f(x)\cdot(x-1)dx\right)^2$$

$$\int_0^1 f^2(x)\,dx\cdot\int_0^1 (x^2-2x+1)\,dx\ge\left(\int_0^1 xf(x)\,dx-\int_0^1 f(x)\,dx\right)^2$$

$$\int_0^1 f^2(x)\,dx\cdot\left(\frac{1^3}{3}-\frac{0^3}{3}-2\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+(1-0)\right)\ge(b-a)^2$$

$$\int_0^1 f^2(x)\,dx\cdot\left(\frac{1}{3}-1+1\right)\ge(b-a)^2$$

$$\int_0^1 f^2(x)\,dx\ge 3(b-a)^2$$

Solution 2 by Marin Chirciu-Romania

Using CBS inequality under integral form:

If $f,g:[a,b]\to\mathbb{R}$, f,g integrable then

$$\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx\ge\left(\int_a^b f(x)\,g(x)dx\right)^2$$

Putting $[a,b]=[0,1]$ and $g(x)=x-1$ we obtain:

$$\int_0^1 f^2(x)\,dx\int_0^1 (x-1)^2\ge\left(\int_0^1 f(x)\,(x-1)dx\right)^2\Rightarrow$$

$$\Rightarrow\int_0^1 f^2(x)\,dx\cdot\frac{1}{3}\ge\left(\int_0^1 xf(x)\,dx-\int_0^1 f(x)\,dx\right)^2\Rightarrow$$

$$\Rightarrow \int_0^1 f^2(x)\,dx \cdot \frac{1}{3} \ge (a-b)^2 \Rightarrow \int_0^1 f^2(x)\,dx \ge 3(a-b)^2.$$

Remark: In the same way.

If $f:[0,1] \to \mathbb{R}, f$ continuous and $\int_0^1 xf(x)\,dx = a, \int_0^1 f(x)\,dx = b, a, b \in \mathbb{R}$ then:

$$\int_0^1 f^2(x)\,dx \ge (3a-2b)^2$$

Marin Chirciu

Solution

Using CBS inequality under integral form:

If $f, g: [a,b] \to \mathbb{R}, f, g$ integrable, then $\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \ge \left(\int_a^b f(x)g(x)\,dx\right)^2$.

Putting $[a,b] = [0,1]$ and $g(x) = 3x-2$ we obtain:

$$\int_0^1 f^2(x)\,dx \int_0^1 (3x-2)^2 \ge \left(\int_0^1 f(x)\,(3x-2)dx\right)^2 \Rightarrow$$

$$\Rightarrow \int_0^1 f^2(x)\,dx \cdot 1 \ge \left(3\int_0^1 xf(x)\,dx - 2\int_0^1 f(x)\,dx\right)^2 \Rightarrow \int_0^1 f^2(x)\,dx \ge (3a-2b)^2.$$