ROMANIAN MATHEMATICAL MAGAZINE

UP.554 We consider the equation: $(1+iz)^{2n} = i\cdot(1+z^2)^n$, where $n\geq 1$ natural number and $i^2=-1$.

a. Prove that the complex number i is a solution of the equation for any $n\geq 1$.

b. Solve the equation in the case $n=1$ and in one of the cases $n=2$ or $n=3$.

c. Find the solution of the equation in the general case $n\in\mathbb{N}^*$

Proposed by Adalbert Kovacs – Romania

Solution 1 by proposer

1. a. By replacing we can check that $z=i$ is a solution of the equation.

1.b. Case $n=1$: We have the equation: $(1+iz)^2 = i\cdot(1+z^2)$, with the roots i and 1.

Case $n=2$. We use the decomposition: $(1+z^2)=(1+iz)(1-iz)$ the given equation becomes: $(1+iz)^4 = i(1+iz)^2\cdot(1-iz)^2 \Leftrightarrow (1+iz)^2\cdot[(1+iz)^2 - i\cdot(1-iz)^2]=0$

$\Leftrightarrow (1+iz)^2=0$ and $(1+iz)^2 = i\cdot(1-iz)^2=0$

The first equation has the double root: i

The second equation: $1+2iz-z^2-i-2z+iz^2=0\Leftrightarrow$

$$\Leftrightarrow (1-i)z^2+2(1-i)z-(1-i)=0\Leftrightarrow$$

$\Leftrightarrow z^2+2z-1=0$, with the roots: $-1+\sqrt{2}$ and $-1-\sqrt{2}$.

Case $n=3$: We have the equation: $(1+i\cdot z)^6 = i\cdot(1+z^2)^3$.

Using the decomposition: $(1+z^2)=(1+iz)(1-iz)$ the equation becomes:

$$(1+iz)^6 = i(1+iz)^3\cdot(1-iz)^3$$

$\Leftrightarrow (1+iz)^3\cdot[(1+iz)^3 - i\cdot(1-iz)^3]=0 \Leftrightarrow (1+iz)^3=0$ or

$$(1+iz)^3 - i\cdot(1-iz)^3 = 0$$

The first equation has a triples solution: $z=1$.

The second equation: $(1+iz)^3+i^3\cdot(1-iz)^3=0\Leftrightarrow$

$$\big(1+iz+i(1-iz)\big)\big(1+iz)^2 - i(1+iz)(1-iz)+i^2\cdot(1-iz)^2\big)=0\Leftrightarrow$$

$1+i+z(1+i)=0$ or $1+2iz-z^2-i-iz^2-1+2iz+z^2=0\Leftrightarrow$

$z+1=0$ or $z^2-4z+1=0$,

with the roots: -1 respectively: $2+\sqrt{3}$ and $2-\sqrt{3}$

So the solutions of the equations are real numbers: $-1, 2+\sqrt{3}, 2-\sqrt{3}$ and the complex number: i, imaginary unit

1.c. General case: We have the equation: $(1+iz)^{2n} = i\cdot(1+z^2)^n \Leftrightarrow$

$$\Leftrightarrow (1+iz)^{2n} = i(1+iz)^n\cdot(1-iz)^n \Leftrightarrow (1+iz)^n[(1+iz)^n - i(1-iz)^n] = 0 \Leftrightarrow$$

$(1+iz)^n = 0$ or $(1+iz)^n - i(1-iz)^n = 0$

From the first equation it follows the solution $z = i$, which is a multiple solution of order n

The second equation can be written: $\left(\frac{1+iz}{1-iz}\right)^n = i$, it is a binomial equation, it goes to the trigonometric form: $\left(\frac{1+iz}{1-iz}\right)^n = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$ and with the extraction of the radical of order n:

$$\frac{1+iz}{1-iz} = \cos\left(\frac{\frac{\pi}{2}+2k\pi}{n}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{n}\right) \Leftrightarrow \frac{1+iz}{1-iz}$$

$$= \cos\left(\frac{\pi+4k\pi}{2n}\right) + i\sin\left(\frac{\pi+4k\pi}{2n}\right)$$

Solving for the z variable, the result is obtained in the form:

$$z = -\frac{1-\cos t - i\cdot\sin t}{i\cdot(1+\cos t + i\cdot\sin t)} = \frac{i\cdot(1-\cos t - i\cdot\sin t)}{1+\cos t + i\cdot\sin t}$$

where $t = \frac{\pi+4k\pi}{2n}$. Using trigonometric formulas and performing the calculations we arrive at the final result: $z = \tan\frac{t}{2}$. We obtain the solutions: $z_k = \tan\left(\frac{\pi+4k\pi}{4n}\right)$, with $k = 0,1,2,\dots,n-1$ (the number of solutions in this form is n). Particular cases:

For $n = 1$ we have the solution: $\tan\frac{\pi}{4} = 1$

For $n = 2$ we have the solutions: $\tan\frac{\pi}{8} = \sqrt{2}+1$ and $\tan\frac{5\pi}{8} = \sqrt{2}-1$

For $n = 3$ we have the solutions: $\tan\frac{\pi}{12} = 2-\sqrt{3}, \tan\frac{5\pi}{12} = 2-\sqrt{3}$ and $\tan\frac{3\pi}{4} = -1$

Solution 2 by Marin Chirciu-Romania

a. It checks $(1+i\cdot i)^{2n} = i\cdot(1+i^2)^n \Leftrightarrow (1-1)^{2n} = i\cdot(1-1)^n \Leftrightarrow 0^{2n} = i\cdot 0^n \Leftrightarrow$

$$\Leftrightarrow 0 = 0$$

b. For $n=1$ we have the equation $(1+iz)^2=i\cdot(1+z^2)\Leftrightarrow(1+i)z^2-2iz+i-1=0$, with $\Delta=4$. We obtain $z_1=2, z+2=2i$.

For $n=2$ we have the equation $(1+iz)^4=i\cdot(1+z^2)^2\Leftrightarrow$

$$\Leftrightarrow(1-i)z^4-4iz^3-(6+2i)z^2+4iz+1-i=0\Leftrightarrow(z-i)^2(z^2+2z-1)=0$$

We obtain $z_1=i, z_{2,3}=-1\pm\sqrt{2}$

For $n=3$ we have the equation $(1+iz)^6=i\cdot(1+z^2)^3\Leftrightarrow$

$$\Leftrightarrow(1+i)z^6-6iz^5+(3i-15)z^4+20iz^3+(3i+15)z^2-6iz+i-1=0$$

$$\Leftrightarrow(z-i)^3(z^3-3z^2-3z+1)=0\Leftrightarrow(z-i)^3(z+1)(z^2-4z+1)=0$$

We obtain $z_1=i, z_2=-1, z_{3,4}=2\pm\sqrt{3}$

c. For solving the equation $(1+iz)^{2n}=i\cdot(1+z^2)^n$ we distinguish the cases:

i. Case 1. $1+z^2\neq 0$

$$(1+iz)^{2n}=i\cdot(1+z^2)^n\Leftrightarrow\left(\frac{1+2iz-z^2}{1+z^2}\right)^n=i, 1+z^2\neq 0$$

Denoting $w=\frac{1+2iz-z^2}{1+z^2}$ we solve the equation $w^n=i\Leftrightarrow w^n=1\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$, with the solutions

$$w_k=\cos\frac{\frac{\pi}{2}+2k\pi}{n}+i\sin\frac{\frac{\pi}{2}+2k\pi}{n}, k=\overline{0,n-1}$$

Going back to the notation we have:

$w=\frac{1+2iz-z^2}{1+z^2}\Leftrightarrow(1+w)z^2-iz+w-1=0,$ with $\Delta=3-4w^2\Rightarrow z=i\pm\sqrt{3-4w^2}$, where

$$w=\cos\frac{\frac{\pi}{2}+2k\pi}{n}+i\sin\frac{\frac{\pi}{2}+2k\pi}{n}, k=\overline{0,n-1}$$

ii. Case 2. $1+z^2=0$

$$1+z^2=0\Leftrightarrow z^2-1\Leftrightarrow z=\pm i$$

The problem is completely solved.