ROMANIAN MATHEMATICAL MAGAZINE

UP.555 Find the solutions of the system:

$$\sqrt{8x+5}+\sqrt{9y+6}=\sqrt{8x+9y+29}$$
$$\sqrt{12x+19}-\sqrt{3y+15}=\sqrt{12x+3y-6}$$

Proposed by Bela Kovacs – Romania

Solution 1 by proposer

We set the conditions of existence: The expressions under the radicals must be positive, both members of the equations must also be positive.Through consecutive squaring we obtain the following system:

$$(8x+5)(9y+6)=81 \text{ and } (12x+19)(3y+15)=400$$
$$\Leftrightarrow (8x+5)(3y+2)=27 \text{ and } (12x+19)(3y+15)=400$$

We do the calculus: $24xy+16x+15y=17$ and $36xy+180x+57y=115$

We multiply with -3, respectively with 2:

$$-72xy-48x-45y=-51 \text{ and } 72xy+360x+114y=230$$

By adding we obtain: $312x+69y=179$

We express the value of y that we substitute in the first equation: $y=\frac{179-312x}{69}$

$$(8x+5)\left(3\cdot\frac{179-312x}{69}+2\right)=27 \Leftrightarrow (8x+5)\left(\frac{179-312x}{23}+2\right)=27 \Leftrightarrow$$
$$(8x+5)(225-312x)=621 \Leftrightarrow (8x+5)(75-104x)=207 \Leftrightarrow$$
$$-832x^2+600x-520x+375-207=0 \Leftrightarrow 832x^2-80x-168=0 \Leftrightarrow$$
$$208x^2-20x-42=0 \Leftrightarrow 104x^2-10x-21=0$$

We solve the quadratic equation obtained: $\Delta_1=25+2184=2209=47^2$

$$x_1=\frac{5+47}{104}=\frac{52}{104}=\frac{1}{2} \text{ and } x_2=\frac{5-47}{104}=\frac{-42}{104}=\frac{-21}{52}$$

We return for the corresponding y values.

$$y_1=\frac{179-156}{69}=\frac{23}{69}=\frac{1}{3} \text{ and } y_2=\frac{179-312\left(-\frac{21}{52}\right)}{69}=\frac{179+126}{69}=\frac{305}{69}$$

By substitution, we find that the first result verifies the system, so it is a solution, and the second result does not verify the system, i.e. in the second equation, a negative number is obtained in the left member, so it is not a solution. So the only system solution is: $\left(\frac{1}{2};\frac{1}{3}\right)$

Solution 2 by Marin Chirciu-Romania

For admissible (x,y)**, the square is successively raised.**

$$\begin{cases}\sqrt{8x+5}+\sqrt{9y+6}=\sqrt{8x+9y+29}\\ \sqrt{12x+19}-\sqrt{3y+15}=\sqrt{12x+3y-6}\end{cases}\Leftrightarrow\begin{cases}\sqrt{8x+5}\cdot\sqrt{9y+6}=9\\ \sqrt{12x+19}\cdot\sqrt{3y+15}=20\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(8x+5)\cdot(9y+6)=81\\ (12x+19)\cdot(3y+15)=400\end{cases}\Leftrightarrow\begin{cases}(8x+5)\cdot(3y+2)=27\\ (12x+19)\cdot(3y+15)=400\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}24xy+16x+15y=17\\ 36xy+180x+57y=115\end{cases}\Leftrightarrow\begin{cases}24xy+16x+15y=17\\ 312x+69y=179\end{cases}$$

Replacing $y=\frac{179-312x}{69}$ **in** $24xy+16x+15y=17$ **we obtain the equation:**

$936x^2-90x-189=0$ **with the solutions** $x_{12}=\frac{45\pm 423}{936}\Rightarrow x_1=\frac{1}{2}$ **and** $x_2=\frac{-21}{52}$

We obtain $(x,y)=\left(\frac{1}{2},\frac{1}{3}\right)$ **also checks** $(x,y)=\left(\frac{-21}{52},\frac{305}{69}\right)$ **doesn't check**

$$\sqrt{12x+19}-\sqrt{3y+15}\geq 0$$

We deduce that $(x,y)=\left(\frac{1}{2},\frac{1}{3}\right)$ **is the unique solution of the system.**