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ABOUT THE PROBLEM 12303-AMM

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Abstract. This paper presents two refinements of an inequality proposed in The American Mathematical
Monthly.

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In The American Mathematical Monthly (AMM), Vol. 129, Nr. 2, February, 2022, was proposed the following problem:

12303. ***Proposed by George Apostolopoulos, Messolonghi, Greece*. Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Prove**

$$\frac{FD}{AB+BC}+\frac{DE}{BC+CA}+\frac{EF}{CA+AB}\le\frac{3}{8}\left(1+\frac{R}{2r}\right).$$

Our purpose is to present two reinforcements of the above inequality.

I. From bisector theorem we have $\frac{BD}{DC}=\frac{c}{b}$, so $BD=\frac{ac}{b+c}$. From cosine law we deduce that

$$FD=\sqrt{BF^2+BD^2-2BF\cdot BD\cdot\cos B}=\sqrt{\left(\frac{ac}{a+b}\right)^2+\left(\frac{ac}{b+c}\right)^2-\frac{2a^2c^2}{(a+b)(b+c)}\cdot\frac{a^2+c^2-b^2}{2ac}}=$$

$$=\sqrt{\frac{abc\cdot(-a^3+b^3-c^3-a^2b+a^2c+ab^2+ac^2+b^2c-bc^2+3abc)}{(a+b)^2(b+c)^2}}.$$

Since $\sqrt{x}+\sqrt{y}+\sqrt{z}\le\sqrt{3(x+y+z)}$, $\forall x,y,z>0$ we get

$$\sum_{cyc}\frac{FD}{AB+BC}=\frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sum_{cyc}\sqrt{-a^3+b^3-c^3-a^2b+a^2c+ab^2+ac^2+b^2c-bc^2+3abc}\le$$

$$\le \frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sqrt{3(-a^3-b^3-c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+9abc}=$$

$$=\frac{\sqrt{abc}}{\prod_{cyc}(a+b)}\sqrt{3(-\sum_{cyc}a^3+\prod_{cyc}(a+b)+7abc)}$$. We denote $2s=a+b+c$.

Since, $\sum_{cyc}ab=s^2+r^2+4Rr$, $\prod_{cyc}(a+b)=\prod_{cyc}(2s-c)=8s^3-2s\cdot 4s^2+\sum_{cyc}ab\cdot 2s-abc=$

$$=\sum_{cyc}ab\cdot 2s-4Rrs=2s(s^2+r^2+4Rr-2Rr)=2s(s^2+2Rr+r^2)$$ and

$\sum_{cyc}a^3=2s(s^2-3r^2-6Rr)$, then by the last inequality we get:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{4Rrs}}{2s(s^2+2Rr+r^2)}\sqrt{3[-2s(s^2-6Rr-3r^2)+2s(s^2+2Rr+r^2)+28Rrs]}=$$

$$=\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}$$. Using Gerretsen inequality, i.e. $s^2\ge 16Rr-5r^2$ we obtain:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{18Rr-4r^2}=\frac{\sqrt{3R(11R+2r)}}{9R-2r}.$$

We will prove that the inequality from above improves the inequality from the problem 12303. Indeed, if we denote $x=R/r$, $x\ge 2$ we have successively that

$$\frac{\sqrt{3R(11R+2r)}}{9R-2r}\le\frac{3}{8}\left(1+\frac{R}{2r}\right)\Leftrightarrow\frac{\sqrt{3x(11x+2)}}{9x-2}\le\frac{3}{8}\left(1+\frac{x}{2}\right)\Leftrightarrow$$

$\Leftrightarrow 3\cdot 256\cdot(11x+2)\le 9(x+2)^2(9x-2)^2\Leftrightarrow 3(x-2)(243x^3+1350x^2+436x-24)\ge 0$, true.

Hence, we obtained the following strengthening of the inequality from AMM:

$$\sum_{cyc}\frac{FD}{AB+BC}\le\frac{\sqrt{12Rr^2(11R+2r)}}{s^2+2Rr+r^2}\le\frac{\sqrt{3R(11R+2r)}}{9R-2r}\le\frac{3}{8}\left(1+\frac{R}{2r}\right), (*).$$

II. Next we will get another reinforcement of inequality from the AMM problem.

Let $s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}}$. By Blundon theorem we know that $s_1 \le s$, so

$$\sum_{cyc} \frac{FD}{AB+BC} \le \frac{\sqrt{12Rr^2(11R+2r)}}{s^2 + 2Rr + r^2} \le \frac{\sqrt{12Rr^2(11R+2r)}}{2R^2 + 12Rr - 2\sqrt{R(R-2r)^3}}.$$

Now, we shall prove that

$$\frac{\sqrt{12Rr^2(11R+2r)}}{2R^2 + 12Rr - 2\sqrt{R(R-2r)^3}} \le \frac{3}{4} \Leftrightarrow \frac{\sqrt{12x(11x+2)}}{2x^2 + 12x - 2\sqrt{x(x-2)^3}} \le \frac{3}{4} \Leftrightarrow$$

$\Leftrightarrow 16 \cdot 12x(11x+2) \le 9\left(2x^2 + 12x - 2\sqrt{x(x-2)^3}\right)^2$, or after some algebra equivalent to

$3x(x-2)\left(3x^2 + 15x + 14 - 3(x+6)\sqrt{x(x-2)}\right) \ge 0$, $\forall x \ge 2$, which is true since

$(3x^2 + 15x + 14)^2 - 9x(x-2)(x+6)^2 \ge 0, \forall x \ge 2 \Leftrightarrow 201x^2 + 1068x + 196 \ge 0, \forall x \ge 2$.

Therefore, we obtain the following refinement

$$\sum_{cyc} \frac{FD}{AB+BC} \le \frac{\sqrt{12Rr^2(11R+2r)}}{s^2 + 2Rr + r^2} \le \frac{\sqrt{12Rr^2(11R+2r)}}{2R^2 + 12Rr - 2\sqrt{R(R-2r)^3}} \le \frac{3}{4} \le \frac{3}{8}\left(1 + \frac{R}{2r}\right), \quad (**).$$