ROMANIAN MATHEMATICAL MAGAZINE

CONNECTIONS BETWEEN FAMOUS CEVIANS-II

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We consider triangle ABC with notations:

p_a, p_b, p_c-Spieker's cevians, n_a, n_b, n_c-Nagel's cevians, g_a, g_b, g_c-Gergonne's cevians

1.) $n_a m_a \geq p_a^2$ (and analogs) [1]

2.) $n_a g_a \geq m_a l_a$(and analogs) [2]

3.) $m_a^2 \geq l_a p_a$(and analogs) [3]

From 1) and 3) we obtain:

$$m_a^3 n_a \geq p_a^3 l_a \text{(and analogs) (1)}$$

We obtain:

$$\sqrt[3]{\frac{n_a}{l_a}} \geq \frac{p_a}{m_a} \text{ (and analogs) (2)}$$

From (2) after summation we obtain:

$$\sum m_a \sqrt[3]{\frac{n_a}{l_a}} \geq p_a+p_b+p_c \text{ (3)}$$

$$\sum \sqrt[3]{\frac{n_a}{l_a}} \geq \sum \frac{p_a}{m_a} \text{ (4)}$$

$$\sum m_a^3 n_a \geq \sum p_a^3 l_a \text{ (5)}$$

From 2) and 3) $\rightarrow$

$$n_a g_a m_a \geq l_a^2 p_a \text{ (and analogs) (6)}$$

From (6) after summation we obtain:

$$\sum n_a g_a m_a \geq \sum l_a^2 p_a \text{ (7)}$$

$$\sum \sqrt{n_a g_a m_a} \geq \sum l_a \sqrt{p_a} \text{ (8)}$$

From (6) and $l_a = 2\frac{\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs)

$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \geq 2\sqrt{\frac{p_a r_b r_c}{n_a g_a m_a}} \text{ (and analogs) (9)}$$

From: $a\,h_a = b\,h_b = c\,h_c$ and (9) we obtain:

$$\sqrt{\frac{h_b}{h_c}}+\sqrt{\frac{h_c}{h_b}} \ge 2\sqrt{\frac{p_a r_b r_c}{n_a g_a m_a}} \text{ (and analogs) (10)}$$

From (9) we obtain:

$$\frac{b}{c}+\frac{c}{b} \ge 2\left(2\frac{p_a r_b r_c}{n_a g_a m_a}-1\right) \text{ (and analogs) (11)}$$

ω=Brocard angle, $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs)(Traian Lalescu)[4] and using (11):

$$\frac{\sin(A+\omega)}{\sin\omega} \ge 2\left(2\frac{p_a r_b r_c}{n_a g_a m_a}-1\right) \text{(and analogs) (12)}$$

$m_a \ge \frac{b^2+c^2}{4R}$ (Tereshin) and bc=2Rh_a (and analogs) $\rightarrow 2\frac{m_a}{h_a} \ge \frac{b}{c}+\frac{c}{b}$(and analogs) and (11) we obtain: $2\frac{m_a}{h_a} \ge 2\left(2\frac{p_a r_b r_c}{n_a g_a m_a}-1\right)$

$$\frac{m_a}{h_a} \ge 2\frac{p_a r_b r_c}{n_a g_a m_a}-1 \text{(and analogs) (13)}$$

We proved: $n_a \ge p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) [5] and using (2) we obtain:

$$n_a\sqrt{\frac{g_a}{l_a}}+m_a\sqrt[3]{\frac{n_a}{l_a}} \ge 2p_a \text{ (and analogs)(14)}$$

From (14) after summation we obtain:

$$\sum\left(n_a\sqrt{\frac{g_a}{l_a}}+m_a\sqrt[3]{\frac{n_a}{l_a}}\right) \ge 2(p_a+p_b+p_c) \text{ (15)}$$

From (2) and (6) after summation we obtain:

$$m_a\left(\sqrt[3]{\frac{n_a}{l_a}}+\frac{n_a g_a}{l_a^2}\right) \ge 2p_a \text{(and analogs) (16)}$$

From (2) and $n_a \ge p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) we obtain:

$$n_a\left(\sqrt{\frac{g_a}{l_a}}+\frac{g_a m_a}{l_a^2}\right) \ge 2p_a \text{(and analogs) (17)}$$

From (2) and $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ we obtain:

$$a^2+b^2+c^2 \ge \frac{4}{3}\sum p_a^2\sqrt[3]{\left(\frac{l_a}{n_a}\right)^2} \text{(18)}$$

From 1.) and $m_a l_a \geq r_b r_c$(and analogs)(Panaitopol) we obtain:

$$\sqrt{\frac{n_a l_a}{r_b r_c}} \geq \frac{p_a}{m_a} \text{ (and analogs) (19)}$$

From (19) after summation we obtain:

$$\sum \sqrt{\frac{n_a l_a}{r_b r_c}} \geq \sum \frac{p_a}{m_a} (20)$$

$$\sum m_a \sqrt{\frac{n_a l_a}{r_b r_c}} \geq p_a + p_b + p_c \text{ (21)}$$

From (2) and (19) we obtain:

$$m_a \left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}} \right) \geq 2p_a \text{(and analogs)(22)}$$

From (22) we obtain:

$$\prod \left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}} \right) \geq 8 \frac{p_a p_b p_c}{m_a m_b m_c} (23)$$

From 1.) and $m_a l_a \geq r_b r_c$(and analogs)(Panaitopol) we obtain:

$$m_a(n_a + l_a) \geq p_a^2 + r_b r_c \text{(and analogs) (24)}$$

From (24) after summation we obtain:

$$\sum m_a(n_a + l_a) \geq p_a^2 + p_b^2 + p_c^2 + p^2 (25)$$

$$m_a + m_b + m_c \geq \sum \frac{p_a^2 + r_b r_c}{n_a + l_a} \text{ (26)}$$

From (22) after summation we obtain:

$$\sum \frac{m_a}{p_a} \left(\sqrt[3]{\frac{n_a}{l_a}} + \sqrt{\frac{n_a l_a}{r_b r_c}} \right) \geq 6 \text{ (27)}$$

From (6) after summation we obtain:

$$m_a + m_b + m_c \geq \sum \frac{l_a^2 p_a}{n_a g_a} \text{ (28)}$$

From (26) and (28) we obtain:

$$m_a + m_b + m_c \geq \sqrt{\sum \frac{l_a^2 p_a}{n_a g_a} \sum \frac{p_a^2 + r_b r_c}{n_a + l_a}} \text{ (29)}$$

From (3) and (21) we obtain:

$$\sqrt{\sum \mathbf{m_a} \sqrt[3]{\frac{\mathbf{n_a}}{\mathbf{l_a}}} \sum \mathbf{m_a} \sqrt{\frac{\mathbf{n_a l_a}}{\mathbf{r_b r_c}}}} \geq \mathbf{p_a}+\mathbf{p_b}+\mathbf{p_c} \ \mathbf{(30)}$$

From (6) after summation we obtain:

$$n_a + n_b + n_c \geq \sum \frac{l_a^2 p_a}{g_a m_a} \ (31)$$

From (31) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) we obtain:

$$\mathbf{n_a + n_b + n_c \geq \sqrt{\left(p_a\sqrt{\frac{l_a}{g_a}} + p_b\sqrt{\frac{l_b}{g_b}} + p_c\sqrt{\frac{l_c}{g_c}}\right)\sum \frac{l_a^2 p_a}{g_a m_a}}} \ \mathbf{(32)}$$

From (6) and $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) after summation we obtain:

$$\mathbf{2n_a \geq p_a\sqrt{\frac{l_a}{g_a}}\left(1 + \frac{l_a}{m_a}\sqrt{\frac{l_a}{g_a}}\right)} \ \textbf{(and analogs) (33)}$$

References

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