ROMANIAN MATHEMATICAL MAGAZINE

CONNECTIONS BETWEEN FAMOUS CEVIANS-III

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In Δ ABC we have:

m_a^2=p(p-a)$+\frac{1}{4}(b-c)^2$ (and analogs)

l_a^2=p(p-a)$-\frac{p(p-a)}{(b+c)^2}(b-c)^2$ (and analogs) [1]

p_a^2= p(p-a) $+p\frac{(3p+a)}{(2p+a)^2}(b-c)^2$(and analogs) [2]

Will prove that: $l_a^2+p_a^2\le 2m_a^2$(and analogs)

After we simplify with 2p(p-a) we obtain:

$(b-c)^2\left[p\frac{(3p+a)}{(2p+a)^2}-\frac{p(p-a)}{(b+c)^2}\right]\le\frac{1}{2}(b-c)^2$

If b=c we obtain equality.

If b≠c we obtain: $p\frac{(3p+a)}{(2p+a)^2}-\frac{p(p-a)}{(b+c)^2}<\frac{1}{2}$

$p\frac{(3p+a)}{(2p+a)^2}=\frac{1}{4}+\frac{8p^2-a^2}{4(2p+a)^2}\rightarrow\frac{1}{4}+\frac{8p^2-a^2}{4(2p+a)^2}<\frac{1}{2}\rightarrow\frac{8p^2-a^2}{4(2p+a)^2}<\frac{1}{4}+\frac{p(p-a)}{(b+c)^2}$

$8p^2-a^2<(2p+a)^2$+p(p-a)$\left(\frac{4p+2a}{b+c}\right)^2$=$(2p+a)^2+$ p(p-a) $\left(\frac{2a+2(b+c)+2a}{b+c}\right)^2$

$8p^2-a^2<(2p+a)^2+$ p(p-a) $\left(2+\frac{4a}{b+c}\right)^2$→TRUE !!!

We obtain:

$$\mathbf{l_a^2+p_a^2\le 2m_a^2}\textbf{(and analogs) (1)}$$

$l_a^2+p_a^2\ge 2p_al_a$ (and analogs) →

$$\mathbf{m_a^2\ge p_al_a(\text{and analogs})(2)}$$

From (1) and (2) after summation we obtain:

$$\mathbf{2m_a\ge p_a+l_a}\textbf{ (and analogs) (3)}$$

From (1) and $4m_a^2=n_a^2+g_a^2+2r_br_c$ (and analogs) [3]

$$\mathbf{n_a^2+g_a^2\ge 2(\,l_a^2+p_a^2-r_br_c)(\text{and analogs})}\textbf{ (4)}$$

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From $(b-c)^2 = n_a^2 + g_a^2 - 2r_br_c$ (and analogs) [3]

We obtain:

$$\mathbf{(b-c)^2 \geq 2(l_a^2 + p_a^2 - 2r_br_c)(and\ analogs)(5)}$$

From (5) and $l_a^2 + p_a^2 \geq 2p_al_a$ (and analogs) we obtain:

$$\mathbf{(b-c)^2 \geq 4(p_al_a - r_br_c)(and\ analogs)(6)}$$

From $p_a \geq m_a$(and analogs)

$m_al_a \geq r_br_c$(and analogs) (Panaitopol)

we obtain:

$$\mathbf{(b-c)^2 \geq 2(l_a^2 + p_a^2 - 2m_al_a)\ (and\ analogs)\ (7)}$$

$$\mathbf{|b\text{-}c| \geq \sqrt{2}\,(p_a - l_a)\ (and\ analogs)\ (8)}$$

We know that $n_a + g_a \geq 2m_a$ (and analogs) [4] and using (3) we obtain:

$$\mathbf{n_a + g_a \geq 2m_a \geq p_a + l_a (and\ analogs)\ (9)}$$

From $|b\text{-}c| \geq n_a - g_a$ (and analogs) [4] and (8) after summation we obtain:

$$\mathbf{2|\ b\text{-}c| \geq n_a + \sqrt{2}\,(p_a - l_a) - g_a (and\ analogs)\ (10)}$$

We know that $2\sum |b-c| = 4[\max(a,b,c) - \min(a,b,c)]$ and using (10) we obtain:

$$\mathbf{\max(a,b,c) - \min(a,b,c) \geq \frac{1}{4}\sum\left(n_a + \sqrt{2}\,(p_a - l_a) - g_a\right)(11)}$$

From $\sum \frac{m_a^2}{h_a^2} = 1 + \frac{1}{2\sin^2\omega}$ and (1) we obtain:

$$\mathbf{1 + \frac{1}{2\sin^2\omega} \geq \frac{1}{2}\sum\frac{l_a^2 + p_a^2}{h_a^2}\ (12)}$$

$$2 \geq \sum\frac{l_a^2 + p_a^2}{h_a^2} - \frac{1}{\sin^2\omega}$$

From $2m_a = \sqrt{2(b^2 + c^2) - a^2}$ (and analogs) and (3)

$$\mathbf{\sqrt{2(b^2 + c^2) - a^2} \geq p_a + l_a\ (and\ analogs)\ (13)}$$

From $2\frac{m_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{b}{c} + \frac{c}{b}$ (and analogs) [5] and (3) we obtain

$$\frac{b}{c}+\frac{c}{b}\geq\frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R}\text{ (and analogs) (14)}$$

$\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) (Traian Lalescu) , $\omega=$ Brocard angle in triangle ABC

$$\frac{\sin(A+\omega)}{\sin\omega}\geq\frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R}\text{ (and analogs)(15)}$$

From (15) we obtain:

$$\frac{1}{\sin\omega}\geq\frac{p_a+l_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R}\text{ (and analogs)(16)}$$

From (10) and $n_a\geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogs) [6] we obtain:

$$2|\,b\text{-}c\,|\geq p_a\sqrt{\frac{l_a}{g_a}}+\sqrt{2}\,(p_a-l_a)-g_a\text{ (and analogs) (17)}$$

From (1) and $n_ag_a\geq m_al_a$ (and analogs) [1] we obtain:

$$n_a\geq\frac{l_a}{g_a}\sqrt{\frac{l_a^2+p_a^2}{2}}\text{ (and analogs) (18)}$$

From (18) after summation we obtain:

$$n_a+n_b+n_c\geq\sum\frac{l_a}{g_a}\sqrt{\frac{l_a^2+p_a^2}{2}}\text{ (19)}$$

From (1) and $m_an_a\geq p_a^2$ (and analogs) [7] we obtain:

$$2n_a\geq\frac{(l_a^2+p_a^2)}{m_a^3}p_a^2\text{ (and analogs)(20)}$$

From (20) after summation we obtain:

$$2(n_a+n_b+n_c)\ \geq\sum\frac{(l_a^2+p_a^2)}{m_a^3}p_a^2\text{(21)}$$

From (20) we obtain:

$$m_a\geq\sqrt[3]{(l_a^2+p_a^2)\ \frac{p_a^2}{2n_a}}\text{ (22)}$$

After summation from (22) we obtain:

$$\mathbf{m_a + m_b + m_c \geq \sum \sqrt[3]{(l_a^2 + p_a^2)\ \frac{p_a^2}{2n_a}}}\ \textbf{(23)}$$

From (22) we obtain: $m_a^2 \geq \sqrt[3]{\left((l_a^2 + p_a^2)\ \frac{p_a^2}{2n_a}\right)^2}$

$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ and after summation we obtain:

$$\boldsymbol{a^2 + b^2 + c^2 \geq \frac{4}{3}\sum \sqrt[3]{\left((l_a^2 + p_a^2)\ \frac{p_a^2}{2n_a}\right)^2}}\ \textbf{(24)}$$

From (20) we obtain $2n_a \frac{m_a^3}{p_a^3} \geq p_a + \frac{l_a^2}{p_a}$ (and analogs) and after summation we obtain:

$$\mathbf{2\sum n_a \frac{m_a^3}{p_a^3} \geq p_a + p_b + p_c + \sum \frac{l_a^2}{p_a}}\ \textbf{(25)}$$

From (25) and using Bergstrom inequality we obtain:

$$\mathbf{2\sum n_a \frac{m_a^3}{p_a^3} \geq p_a + p_b + p_c + \frac{(l_a + l_b + l_c)^2}{p_a + p_b + p_c}}\ \textbf{(26)}$$

From $2n_a \frac{m_a^3}{p_a^3} \geq p_a + \frac{l_a^2}{p_a}$ (and analogs) we obtain:

$$\mathbf{2\frac{n_a p_a}{l_a^2 + p_a^2} \geq \frac{p_a^3}{m_a^3}}\ \textbf{(and analogs) (27)}$$

From (27) after summation we obtain:

$$\mathbf{2\sum \frac{n_a p_a}{l_a^2 + p_a^2} \geq \sum \frac{p_a^3}{m_a^3}}\ \textbf{(28)}$$

$$\mathbf{\sum \sqrt[3]{2\frac{n_a p_a}{l_a^2 + p_a^2}} \geq \frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c}}\ \textbf{(29)}$$

$$\mathbf{\sum m_a \sqrt[3]{2\frac{n_a p_a}{l_a^2 + p_a^2}} \geq p_a + p_b + p_c}\ (30)$$

From (27) and $p_a \geq m_a$ we obtain:

$$\mathbf{2n_a p_a \geq l_a^2 + p_a^2}\ \textbf{(and analogs) (31)}$$

$p_a(2n_a - p_a) \geq l_a^2$ (and analogs) and after summation we obtain:

$$\mathbf{\sum \sqrt{p_a(2n_a - p_a)} \geq l_a + l_b + l_c}\ \textbf{(32)}$$

From $p_a(2n_a - p_a) \geq l_a^2$ (and analogs) and Bergstrom inequality we obtain:

$$\mathbf{p_a + p_b + p_c \geq \frac{(l_a+l_b+l_c)^2}{2(n_a+n_b+n_c)-(p_a+p_b+p_c)}} \quad \textbf{(33)}$$

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