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CONNECTIONS BETWEEN FAMOUS CEVIANS

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We consider the triangle ABC with notations:

p_a, p_b, p_c-Spieker's cevians, n_a, n_b, n_c-Nagel's cevians, g_a, g_b, g_c-Gergonne's cevians

It is known that:

$$\mathbf{n_a m_a \geq p_a^2} \textbf{ (and analogs) [1]}$$

$$\mathbf{n_a g_a \geq m_a l_a} \textbf{(and analogs) [2]}$$

We will find an interesting connection between n_a, g_a, p_a, l_a:

From those relations we obtain a new one:

$$\mathbf{n_a\sqrt{g_a} \geq p_a\sqrt{l_a}} \textbf{ (and analogs) (1)}$$

(1) Is a refinement for $n_a \geq p_a$ because $l_a \geq g_a$.

$$\mathbf{\frac{a}{2r} = \frac{n_a}{h_a} + 2\frac{r_a}{n_a+p}} \textbf{ (and analogs) [3],}$$

and using (1) we obtain:

$$\mathbf{\frac{a}{2r} \geq \frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}} + 2\frac{r_a}{n_a+p}} \text{ (and analogs)(2)}$$

From (2) after summation:

$$\mathbf{\frac{p}{r} \geq \sum \frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}} + 2\sum\frac{r_a}{n_a+p}} \textbf{ (3)}$$

From (1):

$$\mathbf{\sum n_a\sqrt{g_a} \geq \sum p_a\sqrt{l_a}} \textbf{ (4)}$$

$$\mathbf{\sum \frac{n_a}{p_a} \geq \sum\sqrt{\frac{l_a}{g_a}}} \textbf{ (5)}$$

$$\mathbf{n_a + n_b + n_c \geq p_a\sqrt{\frac{l_a}{g_a}} + p_b\sqrt{\frac{l_b}{g_b}} + p_c\sqrt{\frac{l_c}{g_c}}} \textbf{ (6)}$$

a=2Rsin A (Sine Theorem) $\rightarrow \frac{a}{2r} = \frac{R}{r}\sin A$ (and analogs)

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \left(\frac{\mathbf{p_a}}{\mathbf{h_a}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + 2\frac{\mathbf{r_a}}{\mathbf{n_a+p}}\right)\frac{1}{\sin A}\textbf{(and analogs) (7)}$$

From $\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) [4] $\rightarrow 2\frac{r_a}{n_a+p} = \frac{a-\sqrt{4r^2+(b-c)^2}}{2r}$(and analogs)

$$\rightarrow \left(\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \mathbf{p}\right)\left(\mathbf{a} - \sqrt{\mathbf{4r^2 + (b-c)^2}}\right) \leq \mathbf{4rr_a}\textbf{(and analogs) (8)}$$

From $4rr_a = 4(p-b)(p-c)$(and analogs) and (8) $\rightarrow$

$$\left(\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \mathbf{p}\right)\left(\mathbf{a} - \sqrt{\mathbf{4r^2 + (b-c)^2}}\right) \leq \mathbf{4(p-b)(p-c)}\textbf{(9)}$$

From (8) and $r_a + r_b + r_c = 4R + r$ after summation we obtain:

$$\sum\left(\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \mathbf{p}\right)\left(\mathbf{a} - \sqrt{\mathbf{4r^2 + (b-c)^2}}\right) \leq \mathbf{4r(r_a + r_b + r_c) = 4r(4R + r)(10)}$$

From $\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} = \frac{h_a}{r}$ (and analogs)[4]; $\frac{h_a}{r} = 1 + \frac{b+c}{a}$(and analogs) and (1):

$$\frac{\mathbf{h_a}}{\mathbf{r}} \geq \frac{\mathbf{2p_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}\,\textbf{(and analogs)(11)}$$

From (11) after summation:

$$\frac{\mathbf{h_a+h_b+h_c}}{\mathbf{2r}} \geq \sum\frac{\mathbf{p_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}\,\textbf{(12)}$$

From (11) $\rightarrow$

$$\frac{\mathbf{b+c}}{\mathbf{a}} \geq \frac{\mathbf{2p_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} - \mathbf{1}\ \textbf{(and analogs) (13)}$$

From $l_a = \frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs) $\rightarrow l_a l_b l_c = \frac{8abc}{(a+b)(b+c)(c+a)} r_a r_b r_c$

$$\mathbf{8}\frac{\mathbf{r_a r_b r_c}}{\mathbf{l_a l_b l_c}} = \frac{\mathbf{(a+b)(b+c)(c+a)}}{\mathbf{abc}} \geq \prod\left(\frac{\mathbf{2p_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} - \mathbf{1}\right)\textbf{(14)}$$

We know: $\frac{R}{r} - 1 = \frac{n_a^2 + r_a^2}{2h_a r_a}$ (and analogs)[5] we obtain:

$2r_a h_a\left(\frac{R}{r}-1\right)=n_a^2+r_a^2$ (and analogs) $\rightarrow 2\frac{h_a}{n_a}\left(\frac{R}{r}-1\right)=\frac{n_a}{r_a}+\frac{r_a}{n_a}$ (and analogs) using (1) we obtain:

$$\mathbf{2\left(\frac{R}{r}-1\right)\geq\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)(15)}$$

From: $\frac{R}{r}\geq 1+\frac{n_a}{h_a}$(and analogs) $\rightarrow$

$$\mathbf{\frac{R}{r}\geq 1+\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}\text{(and analogs) (16)}}$$

From $\frac{R}{r}\geq 1+\frac{n_a}{h_a}$(and analogs) $\rightarrow\left(\frac{R}{r}-1\right)^3\geq\frac{n_a n_b n_c}{h_a h_b h_c}\rightarrow\frac{R}{r}\geq 1+\sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}}$ we obtain: $\mathbf{\frac{R}{r}\geq 1+}$

$$\mathbf{\sqrt[3]{\frac{p_a}{h_a}\frac{p_b}{h_b}\frac{p_c}{h_c}\sqrt{\frac{l_a}{g_a}\frac{l_b}{g_b}\frac{l_c}{g_c}}}\ (17)}$$

From $p^2=n_a^2+2h_a r_a$ (and analogs) $\rightarrow p^2-n_a^2=2h_a r_a$

$(p-n_a)(p+n_a)=2h_a r_a$, and $\frac{p}{h_a}=\frac{a}{2r}$ (and analogs) $\rightarrow\frac{a}{2r}+\frac{n_a}{h_a}=\frac{2r_a}{p-n_a}$ (and analogs). Using $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) and $\frac{a}{2r}+\frac{n_a}{h_a}=\frac{2r_a}{p-n_a}$ we obtain:

$\frac{2r_a}{p-n_a}=\frac{a+\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs)

$\frac{4r_a r}{p-n_a}=a+\sqrt{4r^2+(b-c)^2}$ (and analogs)

$p=n_a+\frac{4r_a r}{a+\sqrt{4r^2+(b-c)^2}}$ (and analogs)(*)

$p=n_a+\frac{4(p-b)(p-c)}{a+\sqrt{4r^2+(b-c)^2}}$ (and analogs) (**)

From those we obtain:

$$\mathbf{p\geq p_a\sqrt{\frac{l_a}{g_a}}+\frac{4(p-b)(p-c)}{a+\sqrt{4r^2+(b-c)^2}}\ (\text{and analogs})(18)}$$

From (18) after summation we obtain:

$$\mathbf{3p\geq\sum\left(p_a\sqrt{\frac{l_a}{g_a}}+\frac{4(p-b)(p-c)}{a+\sqrt{4r^2+(b-c)^2}}\right)(19)}$$

From $\frac{a}{2r}+\frac{n_a}{h_a}=\frac{2r_a}{p-n_a}$ (and analogs) and (1) we obtain:

$$\frac{2r_a}{p-n_a}\geq\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}+\frac{a}{2r}\text{(and analogs)}(20)$$

From (20) after summation we obtain:

$$\sum\frac{2r_a}{p-n_a}\geq\frac{p}{r}+\sum\frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}}\ \textbf{(21)}$$

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