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FUNDAMENTAL SYMMETRIC PLYNOMIALS IN ALGEBRAIC INEQUALITIES-II

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Abstract: In this article we will present a method for proving symmetric inequalities in two and three variables.

Preliminary considerations: In what follows, we will demonstrate some symmetric inequalities with the help of fundamental symmetric polynomials

Definition. Polinomial $p \in R[x, y]$ **it is symmetrical if** $p(x, y) = p(y, x), \forall x, y \in R$.

Very important to remember: polynomials $\sigma_1 = x + y, \sigma_1 \in R[x, y]$ **and** $\sigma_2 = xy, \sigma_2 \in R[x, y]$ **they are called fundamental symmetric polynomials. Fundamental symmetric polynomials are applied to prove some types of inequalities. The solution of these inequalities is based on Lemma 1.**

Lemma 1. Let $x + y = \sigma_1$; $xy = \sigma_2$. **Both numbers** x, y **are real and positive iff:** $\sigma_1^2 - 4\sigma_2 \geq 0, \sigma_1 \geq 0, \sigma_2 \geq 0$.

Proof. x, y are the roots of equation $z^2 - \sigma_1 z + \sigma_2 = 0$, i.e.

$$x = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}; y = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2} \text{ or } x = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2}; y = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}.$$

x, y are real if $\sigma_1^2 - 4\sigma_2 \geq 0$. If $\sigma_1^2 - 4\sigma_2 = 0$, then $x = y$. So, there is $z > 0$, such that $\sigma_1^2 - 4\sigma_2 = z$. We have $\sigma_1^2 = 4\sigma_2 + z$ or $\sigma_2 = \frac{\sigma_1^2 - z}{4}$, $x \geq 0, y \geq 0$, hence: $\sigma_1 \geq 0, \sigma_2 \geq 0$.

Reciprocal: If $\sigma_1^2 - 4\sigma_2 \geq 0$, then $x, y \in R$ and if $\sigma_1 \geq 0, \sigma_2 \geq 0$, then $x \geq 0$ and $y \geq 0$.

Indeed, if $x < 0$, then $z^2 - \sigma_1 z + \sigma_2 > 0$ for $z = x$ and also $z^2 - \sigma_1 z + \sigma_2 > 0$ for $z = y$. So, $x \geq 0$ and $y \geq 0$.

Lemma 2. We denote $S_k = x^k + y^k$, $\forall k \in N$. **We have:**

$$S_{k+1} = \sigma_1 S_k - \sigma_2 S_{k-1}, \ (S_0 = 2, S_1 = \sigma_1), \forall k \geq 1, \textbf{(i)}.$$

Proof. By some algebra we obtain:

$$x^{k+1}+y^{k+1}=(x+y)(x^k+y^k)-xy(x^{k-1}+y^{k-1})=x^{k+1}+y^{k+1}+xy^k+yx^k-x^ky-xy^k=$$

$$=x^{k+1}+y^{k+1}\text{, q.e.d.}$$

By (i) we deduce Waring's formula:

$$\frac{1}{k}S_k=\sum_m\frac{(-1)^m(k-m-1)!}{m!(k-2m)!}\sigma_1^{k-2m}\sigma_2^m\text{, (ii).}$$

By (ii) we compute S_k ($k\in\{3,4,5,6\}$) and we obtain:

$$S_3=\sigma_1^3-3\sigma_1\sigma_2;\ S_4=\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2;S_5=\sigma_1^5-5\sigma_1^3\sigma_2+5\sigma_1\sigma_2^2;$$

$$S_6=\sigma_1^6-6\sigma_1^4\sigma_2+9\sigma_1^2\sigma_2^2-2\sigma_2^3.$$

Theorem. Any symmetric polynomial $p\in R[x,y]$ **can be expressed uniquely with the help of fundamental symmetric polynomials**.

Proof. We will use (ii) by Lemma 2 and the facts: $ax^ky^y=a\sigma_2^k$ or

$$b(x^ky^l+x^ly^k)=bx^ky^k(x^{l-k}+y^{l-k}),\ l\ge k \text{ şi } b(x^ky^l+x^ly^k)=bx^ly^l(x^{k-l}+y^{k-l}), k\ge l\text{, (iii).}$$

By example: $p=xy(x^3+y^3)+x^2y^2+x^3y^3(x+y)=\sigma_2S_3+\sigma_2^2+\sigma_2^3S_1=$

$$=\sigma_2(\sigma_2-3\sigma_1\sigma_2)+\sigma_2^2+\sigma_2^2\sigma_1=\sigma_1^3\sigma_2-3\sigma_1\sigma_2^2+\sigma_2^2+\sigma_1\sigma_2^2=$$

$$=\sigma_1^3\sigma_2-2\sigma_1\sigma_2^2+\sigma_2^2=\sigma_2(\sigma_1^3-2\sigma_1\sigma_2+\sigma_2).$$

In what follows, we will use the above results to prove a number of inequalities in two variables highlighting, for each individual case, the essence of the solution method.

Observation. The theorem can also be extended for all rational algebraic expressions, as will be seen in the following examples.

Applications

1) If a,b **are positive real numbers, then prove that :**

$$\frac{a^3+b^3}{2}\ge\left(\frac{a+b}{2}\right)^3.$$

Solution. Putting $a+b=\sigma_1,\ ab=\sigma_2$, we obatin:

$\frac{a^3+b^3}{2}-\left(\frac{a+b}{2}\right)^3=\frac{S_3}{2}-\frac{\sigma_1^3}{8}=\frac{1}{2}\left(\sigma_1^3-3\sigma_1\sigma_2\right)-\frac{\sigma_1^3}{8}=\frac{3}{8}\sigma_1 z\geq 0$, true because $z\geq 0$.

2) If a,b,c **are positive real numbers such that** $a+b\geq c$**, then prove that:**

$$a^2+b^2\geq\frac{c^2}{2},\ a^4+b^4\geq\frac{c^4}{8},\ a^8+b^8\geq\frac{c^8}{128}.$$

Generalization: $a^{2^n}+b^{2^n}\geq\frac{c^{2^n}}{2^{n+1}}$.

Solution. We let $a+b=\sigma_1$, $ab=\sigma_2$, so we have:

$$S_2=a^2+b^2=\sigma_1^2-2\sigma_2=\sigma_1^2-2\cdot\frac{\sigma_1^2-z}{4}=\frac{\sigma_1^2+z}{2}\geq\frac{\sigma_1^2}{2}\Leftrightarrow a^2+b^2\geq\frac{c^2}{2}.$$

Analogously:

$$a^4+b^4=(a^2)^2+(b^2)^2\geq\frac{1}{2}\left(\frac{c^2}{2}\right)^2=\frac{c^4}{8};\ a^8+b^8=(a^4)^2+(b^4)^2\geq\frac{1}{2}\left(\frac{c^4}{8}\right)^2=\frac{c^8}{128}.$$

By induction yields that $a^{2^n}+b^{2^n}\geq\frac{c^{2^n}}{2^{n+1}}$.

3) If a,b **are positive real numbers, then prove that:**

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\geq\sqrt{a}+\sqrt{b}.$$

Solution. We denote $\sqrt{a}=u,\sqrt{b}=v$ and the inequality to prove becomes

$$\frac{u^2}{v}+\frac{v^2}{u}\geq u+v\Leftrightarrow u^3+v^3\geq uv(u+v).$$

Since, $u^3+v^3-uv(u+v)=S_3-\sigma_2\sigma_1=\sigma_1(\sigma_1^2-4\sigma_2)\geq 0$, true.

4) If a,b **are positive real numbers, then:** $a^4+b^4\geq a^3b+ab^3$.

Solution. $a^4+b^4-a^3b-ab^3=a^4+b^4-ab(a^2+b^2)=S_4-\sigma_2 S_2=$

$$=\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2-\sigma_2(\sigma_1^2-2\sigma_2)=\sigma_1^4-5\sigma_1^2\sigma_2+4\sigma_2^2=\frac{3z\sigma_1^2+z^2}{4}\geq 0,$$

where $z=\sigma_1^2-4\sigma_2\geq 0$.

5) If a,b are positive real numbers, then

$$a^4+2a^3b+2ab^3+b^4\geq 6a^2b^2.$$

Solution. Let $a+b=\sigma_1, ab=\sigma_2, z=\sigma_1^2-4\sigma_2\geq 0$, then:

$$a^4+2a^3b+2ab^3+b^4-6a^2b^2=S_4+2\sigma_2S_2-6\sigma_2^2=\sigma_1^4-2s_1^2\sigma_2-8\sigma_2^2=$$

$$=(z+4\sigma_2)^2-2(z+4\sigma_2)\sigma_2-8\sigma_2^2=z^2+6\sigma_2z\geq 0, \text{ true.}$$

6) (*I.V. Maftei*) If a,b are positive real numbers such that $a+b=1$, then

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq\frac{25}{2}.$$

Generalization. $\left(a+\frac{1}{a}\right)^n+\left(b+\frac{1}{b}\right)^n\geq\frac{5^n}{2^{n-1}}$, $\forall n\in N$.

Solution. $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2-\frac{25}{2}=a^2+b^2+\frac{a^2+b^2}{a^2b^2}-\frac{17}{2}=S_2+\frac{S_2}{\sigma_2^2}-\frac{17}{2}=$

$=\frac{1}{2\sigma_2^2}(-4\sigma_2^3-15\sigma_2^2-4\sigma_2+2)$ and it remains to prove that

$$4\sigma_2^3+15\sigma_2^2+4\sigma_2\leq 2.$$

Since: $\sigma_2\geq 0$ and $z=\sigma_1^2-4\sigma_2\geq 0$, and using $\sigma_1=1$ yields that $0\leq\sigma_2\leq\frac{1}{4}$.

So, $4\sigma_2^3+15\sigma_2^2+4\sigma_2\leq\frac{1}{16}+\frac{15}{16}+1=2$. We have equality for $a=b=2$.

To prove the generalization we consider the convex function $f:[0,\infty)\to R$, $f(x)=x^n$, $n\in N^*$ and by Jensen' s inequality $f(x)+f(y)\geq 2f\left(\frac{x+y}{2}\right)$.

If we take $x = a + \frac{1}{a}, y = b + \frac{1}{b}$, then we obatin

$$\left(a+\frac{1}{a}\right)^n + \left(b+\frac{1}{b}\right)^n \geq 2\left[\frac{1}{2}\left(\sigma_1 + \frac{\sigma_1}{\sigma_2}\right)\right]^n = \frac{1}{2^{n-1}}\left(1+\frac{1}{\sigma_2}\right)^n.$$

Using: $\sigma_1^2 \geq 4\sigma_2 \Leftrightarrow \frac{1}{\sigma_2} \geq 4 \Leftrightarrow \frac{1}{\sigma_2} + 1 \geq 5$, we will obtain $\left(a+\frac{1}{a}\right)^n + \left(b+\frac{1}{b}\right)^n \geq \frac{5^n}{2^{n-1}}$.

7) (*I.V. Maftei*) If a, b are positive real numbers such that $a^2 + b^2 = 1$, then:

$$\frac{1}{1-a} + \frac{1}{1-b} \geq 4 + 2\sqrt{2}.$$

Solution. We have $\frac{1}{1-a} + \frac{1}{1-b} = \frac{2-\sigma_1}{1-\sigma_1+\frac{\sigma_1^2-1}{2}} = \frac{2(2-\sigma_1)}{(\sigma_1-1)^2}$.

From, $\sigma_1^2 - 2\sigma_2 = 1$ şi $\sigma_1^2 \geq 4\sigma_2$ we have $\sigma_2 \leq \frac{1}{2}$ and $\sigma_1 \leq \sqrt{2}$.

Then, $2 - \sigma_1 \geq 2 - \sqrt{2}$ şi $(\sigma_1 - 1)^2 \leq (\sqrt{2}-1)^2$.

Hence, $\frac{2(2-\sigma_1)}{(\sigma_1-1)^2} \geq \frac{2(2-\sqrt{2})}{(\sqrt{2}-1)^2} = 4 + 2\sqrt{2}$.

8) (AoPS, *Daniel Sitaru*) If $a, b > 0$, then prove that

$$9 \leq \left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right).$$

Solution. LHS becomes

$$\frac{2\sqrt{\sigma_2}}{\sigma_1} + \frac{2\sqrt{\sigma_2}}{\sigma_1} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{\sigma_1}{2\sqrt{\sigma_2}} + \frac{4\sigma_2}{\sigma_1^2} + \frac{\sigma_1^2}{4\sigma_2} \geq 6,$$

which yields by AM-GM inequality. RHS becomes

$$2 + \frac{4\sqrt{\sigma_2}}{\sigma_1} + \frac{\sigma_1}{\sqrt{\sigma_2}} + \frac{4\sigma_2}{\sigma_1^2} \leq \frac{7\sigma_1^2}{4\sigma_2},$$

and if we denote $t=\frac{\sigma_1}{\sqrt{\sigma_2}}\geq 2$, then the last inequality becomes

$$7t^4-4t^3-8t^2-16t-16\geq 0 \Leftrightarrow (t-2)(7t^3+10t^2+12t+8)\geq 0,\ \text{true.}$$

9) (Olympiad, URSS, 1984) If $a,b>0,$ **then:**

$$\frac{(a+b)^2}{2}+\frac{a+b}{4}\geq a\sqrt{b}+b\sqrt{a}\,.$$

Solution. Denoting $\sqrt{a}=x,\sqrt{b}=y,x+y=\sigma_1=S,xy=\sigma_2=P$:

$$\frac{(x^2+y^2)^2}{2}+\frac{x^2+y^2}{4}\geq xy(x+y)\Leftrightarrow 2(S^2-2P)^2+S^2-2P\geq 4SP\Leftrightarrow$$

$$\Leftrightarrow 2S^4-8S^2P+8P^2+S^2-2P\geq 4SP\Leftrightarrow 8P^2-(8S^2+4S+2)P+2S^4+S^2\geq 0.$$

We consider the function $f:\left[0,\frac{S^2}{4}\right]\to R,\ f(P)=8P^2-(8S^2+4S+2)P+2S^4+S^2.$

$\Delta=4(S^3+4S^2+4S+1)>0$ and let P_1,P_2 the roots of equation $f(P)=0$.

We prove that $f(P)\geq 0$, so it is suffices to prove

$$P_1>\frac{S^2}{4}\Leftrightarrow\frac{8S^2+4S+2-2\sqrt{16S^3+4S^2+4S+1}}{16}>\frac{S^2}{4}\Leftrightarrow$$

$$\Leftrightarrow (2S^2+2S+1)^2\geq 16S^3+4S^2+4S+1\Leftrightarrow 4S^2(S-1)^2\geq 0.$$

So, $f(P)\geq 0,\ \forall P\in\left[0,\frac{S^2}{4}\right]$, q.e.d.

10) If $x,y,z>0,$ **such that** $x+y+z=1$**, then** $0\leq xy+yz+zx-2xyz\leq\frac{7}{27}$ **(IMO, 1984).**

Solution. Since $x+y+z=1$, so $z=1-\sigma_1$, where $z\in[0,1]$. The inequality becomes:

$$0\leq\sigma_2+\sigma_1(1-\sigma_1)-2\sigma_2(1-\sigma_1)\leq\frac{7}{27}.$$

We consider the function: $f:\left[0,\frac{\sigma_1^2}{4}\right]\to R$, $f(\sigma_2)=25(2\sigma_1-1)\sigma_2+\sigma_1-\sigma_1^2$, σ_1 fixed.

We have $f(0)=\sigma_1(1-\sigma_1)\geq 0$, $f\left(\frac{\sigma_1^2}{4}\right)=\frac{\sigma_1(2\sigma_1^2-5\sigma_1+4)}{4}>0$.

The equality for $\sigma_1=0$ or $\sigma_1=1$, i.e. for $\{x,y,z\}=\{1,0,0\}$ .Also we consider the function:

$$g:\left[0,\frac{\sigma_1^2}{4}\right]\to R,\ g(\sigma_2)=27(2\sigma_1-1)\sigma_2-27\sigma_1(\sigma_1-1)-7,\ g(0)=27\sigma_1^2+27\sigma_1-7\leq 0.$$

Since, $\Delta=-27<0, g\left(\frac{\sigma_1^2}{4}\right)=\frac{1}{4}(3\sigma_1-2)^2(6\sigma_1-7)\leq 0$, so $g(\sigma_2)\geq 0,\forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

Equality occurs for $\sigma_1=\frac{2}{3}$, i.e. $\{x,y,z\}=\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}$.

11) If $x,y,z>0$, such that $x^2+y^2+z^2+2xyz=1$, then:

$$2(xy+yz+zx)\leq x+y+z\ .$$

Solution. Let $x+y=\max\{x+y,y+z,z+x\}$, so

$$\sigma_1^2-2\sigma_2+z^2+2\sigma_2 z=1\Leftrightarrow z=\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2.$$

The inequality becomes:

$$2\sigma_2+2\sigma_1\left(\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2\right)\leq\sigma_1+\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}-\sigma_2$$

$$\Leftrightarrow\sqrt{\sigma_2^2+2\sigma_2+1-\sigma_1^2}(2\sigma_1-1)\leq 2\sigma_1\sigma_2-3\sigma_2+\sigma_1$$

$$\Leftrightarrow(8\sigma_1-8)\sigma_2^2+(4\sigma_1^2-2\sigma_1+2)\sigma_2-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1\leq 0.$$

We consider the function

$$f:\left[0,\frac{\sigma_1^2}{4}\right]\to R,\ f(\sigma_2)=(8\sigma_1-8)\sigma_2^2+(4\sigma_1^2-2\sigma_1+2)\sigma_2-4\sigma_1^2+4\sigma_1^3+2\sigma_1^2-4\sigma_1+1.$$

We have: $1=x^2+y^2+z^2+2xyz\leq(x+y)^2+z^2$ and since $x+y\geq y+z\geq z$ we deduce

$2(xy)^2\geq(x+y)^2+z^2\geq 1\Leftrightarrow x+y\geq\frac{1}{\sqrt{2}}$. Because $x\leq 1,y\leq 1$ we have $\frac{1}{\sqrt{2}}\leq\sigma_1\leq 2$.

We will prove that $f(\sigma_2)\leq 0,\forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

Case 1. $\sigma_1 \in (0,2]$. It is suffices to prove that $f(0) \le 0$ and $f\left(\frac{\sigma_1^2}{4}\right) \le 0$.

$f(0) = -4\sigma_1^2 + 4\sigma_1^3 + 2\sigma_1^2 - 4\sigma_1 + 1$ and we consider the function

$g(\sigma_1) = -4\sigma_1^2 + 4\sigma_1^3 + 2\sigma_1^2 - 4\sigma_1 + 1$ with

$g'(\sigma_1) = 4(-4\sigma_1^3 + 3\sigma_1^2 + \sigma_1 - 1)$, $g''(\sigma_1) = 4(-12\sigma_1^2 + 6\sigma_1 + 1) \le 0$ since $\sigma_1 > 1 > \frac{3+\sqrt{21}}{12}$.

So g' is decreasing on $(1,2]$, i.e. $g'(\sigma_1) \le g'(1) < 0$. Therefore, g is decreasing on $(1,2]$, i.e. $g(\sigma_1) \le g(1) < 0$, hence $f(0) \le 0$.

$$f\left(\frac{\sigma_1^2}{4}\right) = \frac{(\sigma_1+1)(\sigma_1-1)(\sigma_1^2-6\sigma_1+2)}{2} < 0, \text{ since } \sigma_1^2 - 6\sigma_1 + 2 < -3 < 0.$$

Case 2. $\sigma_1 \in \left[\frac{1}{\sqrt{2}}, 1\right]$.

We prove that $\frac{\sigma_1^2}{4} < \sigma_{2V} \Leftrightarrow \frac{\sigma_1^2}{4} < \frac{4\sigma_1^2 - 2\sigma_1 + 2}{16(1-\sigma_1)} \Leftrightarrow 2\sigma_1^2(1-\sigma_1) < 2\sigma_1^3 - \sigma_1 + 1 \Leftrightarrow$

$\Leftrightarrow (2\sigma_1^2 - 1)\sigma_1 + 1 > 0$, true because $2\sigma_1^2 - 1 > 0$.

$$f\left(\frac{\sigma_1^2}{4}\right) = \frac{(\sigma_1+1)(\sigma_1-1)(\sigma_1^2-6\sigma_1+2)}{2} < 0, \text{ since } \sigma_1^2 - 6\sigma_1 + 2 < \frac{5-6\sqrt{2}}{2} < 0.$$

Therefore, $f(\sigma_2) < 0$, $\forall \sigma_2 \in \left[0, \frac{\sigma_1^2}{4}\right]$, then $f(\sigma_2) \le 0, \forall \sigma_2 \in \left[0, \frac{\sigma_1^2}{4}\right]$ şi $\sigma_1 \in \left[\frac{1}{\sqrt{2}}, 2\right]$.

12) If $x, y, z > 0$, then

$$(x+y+z)^3 \ge 27xyz + \frac{x(y+z-2x)^2}{4}.$$

Solution. $\frac{y}{x} + \frac{z}{x} = \sigma_1, \frac{y}{x} \cdot \frac{z}{x} = \sigma_2$. The inequality is equivalent to

$$(1+\sigma_1)^3 \le 27\sigma_2 + \frac{(\sigma_1-2)^2}{4}.$$

Since $\frac{\sigma_1^2}{4} \ge \sigma_2$ it suffices to prove that

$$(1+\sigma_1)^3 \ge \frac{27\sigma_1^2}{4} + \frac{(\sigma_1-2)^2}{4} \Leftrightarrow 4\sigma_1(\sigma_1-2)^2 \ge 0,\ \text{true.}$$

Remark. A stronger inequality it is:

$$(x+y+z)^3 - 27xyz \ge \max\left\{\frac{x(y+z-2x)^2}{4}, \frac{y(x+z-2y)^2}{4}, \frac{z(x+y-2z)^2}{4}\right\}.$$

13) If $x,y,z>0$**, then:**

$$(x+y+z)^3 - 27xyz \ge \frac{(x+y-2z)^2(4x+4y+z)}{4}.$$

Solution. $\frac{y}{x}+\frac{z}{x}=\sigma_1, \frac{y}{x}\cdot\frac{z}{x}=\sigma_2$, and the inequality becomes

$$(1+\sigma_1)^3 - 27\sigma_2 \ge \frac{(\sigma_1-2)^2(4\sigma_1+1)}{4}.$$

Since $\frac{\sigma_1^2}{4} \ge \sigma_2$ it is suffices to prove that

$$(1+\sigma_1)^3 - \frac{27\sigma_1^2}{4} \ge \frac{(\sigma_1-2)^2(4\sigma_1+1)}{4},\ \text{which is true.}$$

Remark. A stronger inequality it is:

$$(x+y+z)^3 - 27xyz \ge$$

$$\ge \max\left\{\frac{(x+y-2z)^2(4x+4y+4z)}{4}, \frac{(y+z-2x)^2(4x+4y+4z)}{4}, \frac{(z+x-2y)^2(4x+4y+4z)}{4}\right\}.$$

14) (*Marius Drăgan*) If $x,y,z>0$**, then** $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge \left(\frac{y+z}{\sqrt{yz}}+1\right)^2$.

Solution. $y+z=\sigma_1, yz=\sigma_2$. The inequality becomes

$$(x+\sigma_1)\left(\frac{1}{x}+\frac{\sigma_1}{\sigma_2}\right)\geq\left(\frac{\sigma_1}{\sqrt{\sigma_2}}+1\right)^2 \Leftrightarrow 1+\frac{\sigma_1 x}{\sigma_2}+\frac{\sigma_1}{x}+\frac{\sigma_1^2}{\sigma_2}\geq\frac{\sigma_1^2}{\sigma_2}+\frac{2\sigma_1}{\sqrt{\sigma_2}}+1\Leftrightarrow$$

$$\Leftrightarrow \sigma_1\left(\frac{x}{\sigma_2}+\frac{1}{x}\right)\geq\frac{2\sigma_1}{\sqrt{\sigma_2}}\Leftrightarrow\sigma_1\left(\frac{x}{\sigma_2}+\frac{1}{x}-\frac{2}{\sqrt{\sigma_2}}\right)\geq 0\Leftrightarrow\sigma_1\left(\frac{\sqrt{x}}{\sqrt{\sigma_2}}-\frac{1}{\sqrt{x}}\right)\geq 0,$$ which is true.

15) If $a,b,c\in R$ with $a+b+c\geq 0$, then $a^3+b^3+c^3\geq 3abc$.

Solution. $b+c=\sigma_1, bc=\sigma_2$. The inequality becomes

$$a^3+\sigma_1^3-3\sigma_2(\sigma_1+a)\geq 0\Leftrightarrow(\sigma_1+a)(\sigma_1^2-a\sigma_1+a^2-3\sigma_2)\geq 0.$$

Since $\sigma_1+a\geq 0$, it is remains to prove that $\sigma_1^2-a\sigma_1+a^2-3\sigma_2\geq 0$ and since $\frac{\sigma_1^2}{4}\geq\sigma_2$ it suffices to show $\sigma_1^2-a\sigma_1+a^2-3\cdot\frac{\sigma_1^2}{4}\geq 0\Leftrightarrow\frac{(\sigma_1-2a)^2}{4}\geq 0$, true.

16) If $a,b,c>0$, then $a^4+b^4+c^4\geq abc(a+b+c)$.

Solution. $\frac{b}{a}+\frac{c}{a}=\sigma_1, \frac{b}{a}\cdot\frac{c}{a}=\sigma_2$. The inequality becomes

$$1+(\sigma_1^2-2\sigma_2)^2-2\sigma_2^2\geq\sigma_2(1+\sigma_1)\Leftrightarrow$$

$$\Leftrightarrow 2\sigma_2^2-(4\sigma_1^2+\sigma_1+1)\sigma_2+\sigma_1^4+1\geq 0.$$

Considering the function: $f:\left[0,\frac{\sigma_1^2}{4}\right]\to R$, $f(\sigma_2)=2\sigma_2^2-(4\sigma_1^2+\sigma_1+1)\sigma_2+\sigma_1^4+1\geq 0.$

$$f(0)=\sigma_1^4+1>0,\ f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1-2)^2(25^2+25+2)}{4}\geq 0$$

Since: $\frac{\sigma_1^2}{4}<\sigma_{2V}=\frac{4\sigma_1^2+\sigma_1+1}{4}$ yields that $f(\sigma_2)\geq 0$, $\forall\sigma_2\in\left[0,\frac{\sigma_1^2}{4}\right]$.

17) (*I.V. Maftei, L. Toderiuc*, IMAC, 2007) If $a,b,c>0$, then

$$8(a^3+b^3+c^3)^2\geq 9(a^2+bc)(b^2+ac)(c^2+ab).$$

Solution. $\frac{b}{a}=x, \frac{c}{a}=y, x^3+y^3=\sigma_1, xy=\sigma_2$. The inequality becomes

$$8(x^3+y^3+1)^2 \ge 9(x^2+y)(y^2+x)(1+xy) \Leftrightarrow 8(\sigma_1+1)^2 \ge 9(\sigma_1+\sigma_2^2+\sigma_2)(\sigma_2+1) \Leftrightarrow$$

$$\Leftrightarrow 9\sigma_2^3+18\sigma_2^2+(9+9\sigma_1)\sigma_2-8\sigma_1^2-7\sigma_1-8 \le 0.$$

Because $x^3+y^3 \ge 2\sqrt{x^3y^3} \Leftrightarrow \sigma_1^2 \ge 4\sigma_2^3$; denoting $u^3=\frac{\sigma_1}{2}$, then $\sigma_2 \le u^2$. So,

$$9\sigma_2^3+18\sigma_2^2+(9+9\sigma_1)\sigma_2-8\sigma_1^2-7\sigma_1-8 \le 9u^6+18u^4+(9-18u^3)u^2-32u^6-14u^3-8.$$

It remains to prove that $23u^6-18u^5-18u^4+14u^3-9u^2+8 \ge 0 \Leftrightarrow$

$\Leftrightarrow (u-1)^2(23u^4+28u^3+15u^2+16u+8) \ge 0$, true.

The equality occurs for $u=1$ and $x=y$, i.e. $x=y=1$, so $a=b=c$.

18) (Moldavian Math Olympiad, 1993) If $x,y,z>0$**, then**

$$x(x-z)^2+y(y-z)^2 \ge (x-z)(y-z)(x+y-z).$$

Solution. $\frac{x}{z}+\frac{y}{z}=\sigma_1, \frac{x}{z}\cdot\frac{y}{z}=\sigma_2$. The inequality becomes

$$\sigma_1(\sigma_1^2-3\sigma_2)+\sigma_1-2(\sigma_1^2-2\sigma_2) \ge (\sigma_2+1-\sigma_1)(\sigma_1-1) \Leftrightarrow$$

$$\Leftrightarrow (5-3\sigma_1)\sigma_2+(\sigma_2+1)(\sigma_1-1)^2 \ge 0.$$

Considering the function $f:\left[0,\frac{\sigma_1^2}{4}\right] \to R$, $f(\sigma_2)=(5-3\sigma_1)\sigma_2+(\sigma_2+1)(\sigma_1-1)^2$, since

$f\left(\frac{\sigma_1^2}{4}\right)=\frac{(\sigma_1-2)^2}{4} \ge 0$, yields $f(\sigma_2) \ge 0$, $\forall \sigma_2 \in \left[0,\frac{\sigma_1^2}{4}\right]$.

Equality iff $\sigma_1=2$, i.e. $x=y=z$.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro