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**J.2341 If $a, b, c, d, e, t, u > 0$, then:**

$$(a^2+t)(b^2+t)(c^2+t)(d^2+u)(e^2+u) \geq \frac{3}{4}t^2u(a+b+c)^2(d+e)^2$$

*Proposed by D.M.Bătinețu-Giurgiu – Romania*

*Solution by Titu Zvonaru-Romania*

It is known the inequality of Arkady Alt (see for example $[1]$):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2 \quad (1)$$

with equality if and only if $x = y = z, t = x\sqrt{2}$. Using $(1)$, we obtain

$$(a^2+t)(b^2+t)(c^2+t) \geq \frac{3}{4}(\sqrt{t})^4(a+b+c)^2 \quad (2)$$

with equality if and only if $a = b = c, t = 2a^2$.

Applying Cauchy-Buniakovski-Schwarz inequality we deduce that

$$(d^2+u)(u+e^2) \geq \left(d\sqrt{u}+e\sqrt{u}\right)^2 = u(d+e)^2 \quad (3)$$

with equality if and only if $\frac{d^2}{u} = \frac{u}{e^2}$ that is $u = de$. Multiplying $(2)$ and $(3)$:

$$(a^2+t)(b^2+t)(c^2+t)(d^2+u)(e^2+u) \geq \frac{3}{4}t^2u(a+b+c)^2(d+e)^2$$

$[1]$ D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024

**ARKADY ALT'S INEQUALITY:**

If $t, x, y, z > 0$ then the following relationship holds:

$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$ with equality if and only if

$$x = y = z = \frac{t}{\sqrt{2}}.$$

Proof: $(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.