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J.2342 In ΔABC the following relationship holds:

$$\frac{m_a+m_b}{h_c^3}+\frac{m_b+m_c}{h_a^3}+\frac{m_c+m_a}{h_b^3}\geq\frac{2\sqrt{3}}{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți – Romania

Solution by Titu Zvonaru-Romania

We have $ah_a=bh_b=ch_c=2F$ and $r_ar_br_c=Fs$. We will use the following inequality

$m_am_bm_c\geq r_ar_br_c$ (item 8.21 from $[1]$),$s^2\geq 3F\sqrt{3}$ (item 4.2 from $[1]$),

Carlitz's inequality $(abc)^{2/3}\geq\frac{4}{3}\sqrt{3}F$, and Cesaro inequality:

$$(a^3+b^3)(b^3+c^3)(c^3+a^3)\geq 8a^3b^3c^3.$$

It follows that:

$$\frac{m_a+m_b}{h_c^3}+\frac{m_b+m_c}{h_a^3}+\frac{m_c+m_a}{h_b^3}\geq\frac{c^3m_a+c^3m_b}{8F^3}+\frac{a^3m_b+a^3m_c}{8F^3}+\frac{b^3m_c+b^3m_a}{8F^3}$$

$$=\frac{1}{8F^3}\left(m_a(b^3+c^3)+m_b(c^3+a^3)+m_c(a^3+b^3)\right)\geq$$

$$\geq\frac{3}{8F^3}\left(m_am_bm_c(a^3+b^3)(b^3+c^3)(c^3+a^3)\right)^{\frac{1}{3}}\geq$$

$$\geq\frac{6}{8F^3}abc(m_am_bm_c)^{\frac{1}{3}}\geq$$

$$\geq\frac{3}{4F^3}\left(2^9(\sqrt{3})^{-\frac{9}{2}+\frac{1}{2}+1}F^{\frac{9}{2}+\frac{1}{2}+1}\right)^{1/3}=\frac{2\sqrt{3}}{F}.$$

Equality holds if and only if the triangle ABC is equilateral.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969