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J.2343 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot ab+\frac{y+z}{x}\cdot bc+\frac{z+x}{y}ca\geq 8\sqrt{3}F.$$

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Solution by Titu Zvonaru-Romania

It is known that $(x+y)(y+z)(z+x)\geq 8xyz$.

Using $AM-GM$ inequality and Carlitz's inequality $(abc)^{2/3}\geq \frac{4}{3}\sqrt{3}F$, we obtain

$$\frac{x+y}{z}\cdot ab+\frac{y+z}{z}\cdot bc+\frac{z+x}{y}\cdot ca\geq 3\left(\frac{(x+y)(y+z)(z+x)}{xyz}\right)^{\frac{1}{3}}(abc)^{\frac{2}{3}}$$
$$\geq 3\left(8(abc)^2\right)^{\frac{1}{3}}=6(abc)^{\frac{2}{3}}\geq 6\cdot\frac{4}{3}\cdot\sqrt{3}\cdot F=8\sqrt{3}\cdot F.$$

Equality holds if and only if $x=y=z$ and the triangle ABC is equilateral.