ROMANIAN MATHEMATICAL MAGAZINE

J.2345 If $m \geq 0, x, y, z > 0$, then

$$(x^{2m+2}+1)(y^{2m+2}+1)(z^{2m+2}+1) > \frac{3^{m+1}}{2^{5m+2}}(x+y+z)^{2m+2}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți - Romania

Solution by Titu Zvonaru-Romania

By Hölder inequality we have

$$\underbrace{(1+1)\dots(1+1)}_{m}(x^{2m+2}+1) \geq \left((x^{2m+2})^{\frac{1}{m+1}}+1\right)^{m+1} = (x^2+1)^{m+1} \quad (1)$$

Using (1) we obtain

$$(x^{2m+2}+1)(y^{2m+2}+1)(z^{2m+2}+1) \geq \frac{(x^2+1)^{m+1}}{2^m}\cdot\frac{(y^2+1)^{m+1}}{2^m}\cdot\frac{(z^2+1)^{m+1}}{2^m} \quad (2)$$

Applying Arqady Alt inequality $(x^2+t)(y^2+t)(z^2+t) \geq \frac{3}{4}t^4(x+y+z)^2$ for $t=1$, it results that

$$(x^2+1)(y^2+1)(z^2+1) \geq \frac{3}{4}(x+y+z)^2 \qquad (3)$$

with equality if and only if $x=y=z=\frac{1}{\sqrt{2}}$.

Using inequalities (2) and (3), it follows that

$$(x^{2m+2}+1)(y^{2m+2}+1)(z^{2m+2}+1) \geq \frac{1}{2^{3m}}\left((x^2+1)(y^2+1)(z^2+1)\right)^{m+1}$$

$$\geq \frac{1}{2^{3m}}\cdot\left(\frac{3}{4}(x+y+z)^2\right)^{m+1} =$$

$$= \frac{1}{2^{3m}}\cdot\frac{3^{m+1}}{2^{2m+2}}(x+y+z)^{2m+2} = \frac{3^{m+1}}{2^{5m+2}}(x+y+z)^{2m+2}.$$

In (1) equality holds if and only if $x=1$, and in (3) equality holds if and only if $x=y=z=\frac{1}{\sqrt{2}}$.

It results that the inequality of the problem is strict.