ROMANIAN MATHEMATICAL MAGAZINE

J.2348 If $x, y > 0$, then in ΔABC holds:

$$x^2(a^2+b^2+c^2)+y^2(r_a^2+r_b^2+r_c^2)\ge 12xyF+(xa-yr_a)^2+(xb-yr_b)^2+(xc-yr_c)^2.$$

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The inequality is equivalent with:

$$xy(ar_a+br_b+cr_c)\ge 6xyF$$

Since $r_a=\frac{F}{s-a}$, we have:

$$ar_a+br_b+cr_c=F\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right).$$

It remains to prove that

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\ge 6\Leftrightarrow$$

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\ge 3.$$

Applying Bergström inequality and the well-known inequality $a^2+b^2+c^2\ge ab+bc+ca$, we obtain

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}=$$

$$=\frac{a^2}{ab+ca-a^2}+\frac{b^2}{bc+ab-b^2}+\frac{c^2}{ca+bc-c^2}\ge$$

$$\ge\frac{(a+b+c)^2}{2(ab+bc+ca)-(a^2+b^2+c^2)}\ge$$

$$\ge\frac{6(ab+bc+ca)-3(a^2+b^2+c^2)}{2(ab+bc+ca)-(a^2+b^2+c^2)}=3.$$

Equality holds if and only if ΔABC s equilateral.