ROMANIAN MATHEMATICAL MAGAZINE

J.2350 If $x,y,z>0$, then in ΔABC holds:

$$\frac{a^3\cdot x^2}{h_a}+\frac{b^3\cdot y^2}{h_b}+\frac{c^3\cdot z^2}{h_c}\geq\frac{8}{3}(xy+yz+zx)\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

We have $ah_a=bh_b=ch_c=2F$.

Applying the known inequality $3(x^2+y^2+z^2)\geq(x+y+z)^2$

and Oppenheim inequality $a^2x+b^2y+c^2z\geq 4F\sqrt{xy+yz+zx}$,

it follows that:

$$\frac{a^3\cdot x^2}{h_a}+\frac{b^3\cdot y^2}{h_b}+\frac{c^3\cdot z^2}{h_c}=\frac{a^4\cdot x^2}{2F}+\frac{b^4\cdot y^2}{2F}+\frac{c^4\cdot z^2}{2F}\geq\frac{(a^2x+b^2y+c^2z)^2}{6F}$$
$$\geq\frac{16F^2(xy+yz+zx)}{6F}=\frac{8}{3}(xy+yz+zx)\cdot F$$

Equality holds if and only if ΔABC is equilateral and $x=y=z$.