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J.2351 If $m \geq 0, x, y, z > 0$ and $T, U \in Int(\Delta ABC), t_a = d(T, BC)$,

$t_b = d(T, CA), t_c = d(T, AB), u_a = d(U, BC), u_b = d(U, CA)$,

$u_c = d(U, AB)$, **then:**

$$\frac{a^{m+2}x^{m+1}}{(t_a+u_a)^m} + \frac{b^{m+2}y^{m+1}}{(u_b+u_b)^m} + \frac{c^{m+2}z^{m+1}}{(t_c+u_c)^m} \geq 4(xy+yz+zx)^{\frac{m+1}{2}} \cdot F$$

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Solution by Titu Zvonaru-Romania

We have $at_a + bt_b + ct_c = 2F, au_a + bu_b + cu_c = 2F$.

Applying Radon's inequality and Oppenheim inequality:

$a^2x + b^2y + c^2z \geq 4F\sqrt{xy+yz+zx}$ it follows that:

$$\frac{a^{m+2}x^{m+1}}{(t_a+u_a)^m} + \frac{b^{m+2}y^{m+1}}{(t_b+u_b)^m} + \frac{c^{m+2}z^{m+1}}{(t_c+u_c)^m} =$$

$$= \frac{a^{2m+2}x^{m+1}}{(at_a+au_a)^m} + \frac{b^{2m+2}y^{m+1}}{(bt_b+bu_b)^m} + \frac{c^{2m+2}z^{m+1}}{(ct_c+cu_c)^m} =$$

$$= \frac{(a^2x)^{m+1}}{(at_a+au_a)^m} + \frac{(b^2y)^{m+1}}{(bt_b+bu_b)^m} + \frac{(c^2z)^{m+1}}{(ct_c+cu_c)^m} \geq$$

$$\geq \frac{(a^2x+b^2y+c^2z)^{m+1}}{(4F)^m} \geq \frac{4^{m+1}(xy+yz+zx)^{\frac{m+1}{2}}}{4^mF^m} = 4(xy+yz+zx)^{\frac{m+1}{2}} \cdot F.$$

Equality holds if and only if ΔABC is equilateral, $x = y = z$ and T, U are the circumcenter.