ROMANIAN MATHEMATICAL MAGAZINE

J.2352 In ΔABC the following relationship holds:

$$(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)\geq 36F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți – Romania

Solution by Titu Zvonaru-Romania

Using Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$ for $t=1$ and Gordon's inequality $ab+bc+ca\geq 4F\sqrt{3}$, it follows that:

$$(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)\geq \frac{3}{4}(ab+bc+ca)^2\geq \frac{3}{4}\left(4F\sqrt{3}\right)^2=36F^2.$$

Equality holds if and only if ΔABC is equilateral with $a^2=\frac{1}{\sqrt{2}}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq \frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.