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J.2353 If $x, y, t > 0$, then in ΔABC holds:

$$\left(\left(xm_a^2+ym_b^2\right)^2+t^2\right)\left(\left(xm_b^2+ym_c^2\right)^2+t^2\right)((xm_c^2+ym_a^2)^2+t^2)\ge\frac{81}{4}t^4(x+y)^2F^2$$

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It is known the inequality of Arqady Alt (see for example $[1]$):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x=y=z, t=x\sqrt{2}$

Using (1), the formula $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$, and Ionescu-Weitzenbock inequality $a^2+b^2+c^2\ge 4\sqrt{3}F$, it follows that

$$\left(\left(xm_a^2+ym_b^2\right)^2+t^2\right)\left(\left(xm_b^2+ym_c^2\right)^2+t^2\right)((xm_c^2+ym_a^2)^2+t^2)$$
$$\ge\frac{3}{4}t^4\left(x\left(m_a^2+m_b^2+m_c^2\right)+y\left(m_a^2+m_b^2+m_c^2\right)\right)^2$$
$$=\frac{3}{4}t^4(x+y)^2\frac{9}{16}(a^2+b^2+c^2)^2\ge\frac{27}{64}t^4(x+y)^2\left(4\sqrt{3}F\right)^2$$
$$=\frac{81}{4}t^4(x+y)^2F^2.$$

Equality holds if and only if ΔABC equilateral and $t=\frac{3a(x+y)}{4}\sqrt{2}$.

$[1]$ D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024

ARKADI ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

It is easy to see that the equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.