J.2354 If $t,x,y,z>0$, then in ΔABC holds:

$$((x^2+a^4)^2+t^2)((y^2+b^4)^2+t^2)((z^2+c^4)^2+t^2)\geq 48(xy+yz+zx)t^4F^2$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

It is known the inequality of Arqady Alt (see for example $[1]$):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x=y=z, t=x\sqrt{2}$.

Using (1), $AM-GM$ inequality, the known inequality $x^2+y^2+z^2\geq xy+yz+zx$, the inequality $a^4+b^4+c^4\geq 16F^2$ (item 4.10 from $[2]$), it follows that

$$((x^2+a^4)^2+t^2)((y^2+b^4)^2+t^2)((z^2+c^4)^2+t^2)\geq$$

$$\geq \frac{3}{4}t^4(x^2+y^2+z^2+a^4+b^4+c^4)^2\geq$$

$$\geq \frac{3}{4}t^4\cdot 4(x^2+y^2+z^2)(a^4+b^4+c^4)\geq 48(xy+yz+zx)t^4F^2.$$

Equality holds if and only if ΔABC is equilateral and $x=y=z$.

$[1]$ D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024

$[2]$ O. Bottema, *Geometric Inequalities*, Groningen 1969

ARKADI ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq \frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.