ROMANIAN MATHEMATICAL MAGAZINE

J.2355 If $x,y,z>0$, then in ΔABC holds:

$$(a^4+x^2)(b^4+y^2)(c^4+z^2)\ge 36\cdot(xyz)^{\frac{4}{3}}\cdot F^2$$

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Applying Arqady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$, $AM-GM$ inequality and Carlitz inequality $(abc)^{2/3}\ge\frac{4}{\sqrt{3}}F$, it follows that

$$(a^4+x^2)(b^4+y^2)(c^4+z^2)=x^2y^2z^2\left(\left(\frac{a^2}{x}\right)^2+1\right)\left(\left(\frac{b^2}{y}\right)^2+1\right)\left(\left(\frac{c^2}{z}\right)^2+1\right)$$

$$\ge\frac{3x^2y^2z^2}{4}\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)^2\ge\frac{3x^2y^2z^2}{4}\left(3\left(\frac{a^2b^2c^2}{xyz}\right)^{\frac{1}{3}}\right)^2$$

$$=\frac{27}{4}\cdot(xyz)^{2-\frac{2}{3}}\cdot(abc)^{\frac{4}{3}}\ge F^2\ge\frac{27}{4}\cdot(xyz)^{\frac{4}{3}}\cdot\left(\frac{4}{\sqrt{3}}F\right)^2=36\cdot(xyz)^{\frac{4}{3}}\cdot F^2.$$

Equality holds if and only if ΔABC is equilateral and $x=y=z=a^2\sqrt{2}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\ge$$

$$\ge\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.