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J.2357 If $x,y>0$ and $T,U\in Int(\Delta ABC), t_a=d(T,BC), t_b=d(T,CA),$

$t_c=d(T,AB), u_a=d(U,BC), u_b=d(U,CA), u_c=d(U,AB)$, then:

$$\frac{a^3b}{(xt_b+yu_b)^2}+\frac{b^3c}{(xt_c+yu_c)^2}+\frac{c^3a}{(xt_a+yu_a)^2}\ge\frac{48\sqrt{3}}{(x+y)^2}\cdot F.$$

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Solution by Titu Zvonaru-Romania

We have $at_a+bt_b+ct_c=2F, au_a+bu_b+cu_c=2F$.

Applying Radon's inequality and Gordon's inequality $ab+bc+ca\ge 4\sqrt{3}F$

it follows that:

$$\frac{a^3b}{(xt_b+yu_b)^2}+\frac{b^3c}{(xt_c+yu_c)^2}+\frac{c^3a}{(xt_a+yu_a)^2}=$$

$$=\frac{a^3b^3}{(xbt_b+ybu_b)^2}+\frac{b^3c^3}{(xct_c+ycu_c)^2}+\frac{c^3a^3}{(xat_a+yau_a)^2}\ge$$

$$\ge\frac{(ab+bc+ca)^3}{\left(x(at_a+bt_b+ct_c)+y(au_a+bu_b+cu_c)\right)^2}\ge\frac{\left(4\sqrt{3}F\right)^3}{(x+y)^2 4F^2}=\frac{48\sqrt{3}}{(x+y)^2}\cdot F.$$

Equality holds if and only if ΔABC is equilateral and T,U are the circumcenter.