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J.2358 If $x, y > 0$ and $U, V \in Int(\Delta ABC), u_a = d(U, BC), u_b = d(U, CA)$,

$u_c = d(U, AB), v_a = d(V, BC), v_b = d(V, CA), v_c = d(V, AB)$, then:

$$\frac{a^3}{xu_a + yv_a} + \frac{b^3}{xu_b + yv_b} + \frac{c^3}{xv_c + yu_c} \geq \frac{24}{x+y} \cdot F.$$

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Solution by Titu Zvonaru-Romania

We have $au_a + bu_b + cu_c = 2F, av_a + bv_b + cv_c = 2F$.

Applying Bergström's inequality and Ionescu-Weitzenbock's inequality

$a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ it follows that:

$$\frac{a^3}{xu_a + yv_a} + \frac{b^3}{xu_b + yv_b} + \frac{c^3}{xv_c + yu_c} = \frac{a^4}{xau_a + yav_a} + \frac{b^4}{xbu_b + ybv_b} + \frac{c^4}{xcv_c + ycu_c} \geq$$

$$\geq \frac{(a^2 + b^2 + c^2)^2}{x(au_a + bu_b + cu_c) + y(av_a + bv_b + cv_c} \geq \frac{48F^2}{2xF + 2yF} = \frac{24}{x+y} \cdot F.$$

Equality holds if and only if ΔABC is equilateral.