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J.2359 If $m \geq 0, x, y > 0$ and $M \in Int(\Delta ABC), d_a = d(M, BC)$, $d_b = d(M, CA), d_c = d(M, AB)$, then:

$$\frac{a^{m+2}}{(xr + yd_a)^m} + \frac{b^{m+2}}{(xr + yd_b)^m} + \frac{c^{m+2}}{(xr + yd_c)^m} \geq \frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}}{(x + y)^m} \cdot F.$$

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We have $ad_a + bd_b + cd_c = 2F, r(a + b + c) = 2F$.

Applying Radon's inequality and Ionescu-Weitzenbock's inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$, it follows that:

$$\frac{a^{m+2}}{(xr + yd_a)^m} + \frac{b^{m+2}}{(xr + yd_b)^m} + \frac{c^{m+2}}{(xr + yd_c)^m} =$$

$$= \frac{a^{2m+2}}{(xar + yad_a)^m} + \frac{b^{2m+2}}{(xbr + ybd_b)^m} + \frac{c^{2m+2}}{(xcr + ycd_c)^m} =$$

$$= \frac{(a^2)^{m+1}}{(xar + yad_a)^m} + \frac{(b^2)^{m+1}}{(xbr + ybd_b)^m} + \frac{(c^2)^{m+1}}{(xcr + ycd_c)^m} \geq$$

$$\geq \frac{(a^2 + b^2 + c^2)^{m+1}}{(xr(a + b + c) + y(ad_a + bd_b + cd_c)^m} \geq$$

$$\geq \frac{\left(4\sqrt{3}F\right)^{m+1}}{(2xF + 2yF)^m} = \frac{2^{2m+2}\left(\sqrt{3}\right)^{m+1}F^{m+1}}{2^m(x + y)^m F^m} = \frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}}{(x + y)^m} \cdot F.$$

Equality holds if and only if ΔABC is equilateral.