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J.2361 If $t > 0$, then in ΔABC holds:

$$(m_a^4+t^2)(m_b^4+t^2)(m_c^4+t^2) \geq \frac{81}{4}t^4F^2$$

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It is known the inequality of Arqady Alt (see for example [1]):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x = y = z, t = x\sqrt{2}$.

Using (1), the formula $m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$, and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2 \geq 4\sqrt{3}F$, it follows that

$$(m_a^4+t^2)(m_b^4+t^2)(m_c^4+t^2) \geq \frac{3}{4}t^4(m_a^2+m_b^2+m_c^2)^2 =$$

$$= \frac{3}{4}t^4 \cdot \frac{9}{16}(a^2+b^2+c^2)^2 \geq \frac{27}{64}t^4(4\sqrt{3}F)^2 = \frac{81}{4}t^4F^2.$$

Equality holds if and only if ΔABC is equilateral and $t = \frac{3a^2\sqrt{2}}{4}$.

[1] **D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024**

ARKADI ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.