ROMANIAN MATHEMATICAL MAGAZINE

J.2362 If $a, b, c, t > 0$, then

$$(a^2+t)(b^2+t)(c^2+t) \geq \frac{9}{4}t^2(ab+bc+ca).$$

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Using Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^2$

and the well known inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, it follows that:

$$(a^2+t)(b^2+t)(c^2+t) = \left(a^2+\left(\sqrt{t}\right)^2\right)\left(b^2+\left(\sqrt{t}\right)^2\right)\left(c^2+\left(\sqrt{t}\right)^2\right) \geq$$

$$\geq \frac{3}{4}\left(\sqrt{t}\right)^4(a+b+c)^2 \geq \frac{9}{4}t^2(ab+bc+ca).$$

Equality holds if and only if $a = b = c = \sqrt{\frac{t}{2}}$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.