

ROMANIAN MATHEMATICAL MAGAZINE

J.2363 If $x, y > 0$, then in $\triangle ABC$ holds:

$$((xr_a + yr_b)^2 + r^2)((xr_b + yr_c)^2 + r^2)((xr_c + yr_a)^2 + r^2) \geq \frac{9}{4}(x + y)^2 r^2 F^2$$

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It is known the inequality of Arkadi Alt (see for example [1]):

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4}t^4(x + y + z)^2 \quad (1)$$

with equality if and only if $x = y = z, t = x\sqrt{2}$.

Since $r_a = \frac{F}{s-a}, F = sr$ and $ab + bc + ca = s^2 + r^2 + 4Rr$ obtain

$$\begin{aligned} r_a + r_b + r_c &= \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} = \\ &= \frac{F(s^2 - s(b+c) + bc + s^2 - s(c+a) + ca + s^2 - s(a+b) + ab)}{(s-a)(s-b)(s-c)} = \\ &= \frac{Fs(3s^2 - 4s^2 + s^2 + r^2 + 4Rr)}{s(s-a)(s-b)(s-c)} = \frac{Fsr(4R+r)}{F^2} = 4R+r \quad (2) \end{aligned}$$

Using (1), (2) and the inequality $r(4R+r) \geq F\sqrt{3}$ (item 7.2 from [2]), it follows that

$$\begin{aligned} &((xr_a + yr_b)^2 + r^2)((xr_b + yr_c)^2 + r^2)((xr_c + yr_a)^2 + r^2) \geq \\ &\geq \frac{3}{4}r^4(x(r_a + r_b + r_c) + y(r_a + r_b + r_c))^2 = \\ &= \frac{3}{4}r^2(x+y)^2(r(4R+r))^2 \geq \frac{9}{4}r^2(x+y)^2F^2. \end{aligned}$$

Equality holds if and only if $\triangle ABC$ equilateral and $r = \frac{a(x+y)\sqrt{6}}{2}$, means that $\triangle ABC$ is equilateral and $x + y = \frac{\sqrt{2}}{6}$.

[1] D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, *Recreații Matematice* nr. 1/2024

[2] O. Bottema, *Geometric Inequalities*, Groningen 1969

ARKADI ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

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$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4}t^4(x + y + z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4}t^2((x + y)^2 + t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x - y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$\begin{aligned}(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) &\geq \frac{3t^2}{4}((x + y)^2 + t^2)(t^2 + z^2) \geq \\ &\geq \frac{3t^2}{4}(t(x + y) + tz)^2 = \frac{3}{4}t^4(x + y + z)^2.\end{aligned}$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.