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J.2364 In ΔABC the following relationship holds:

$$((r_a+r_b)^2+R^2)((r_b+r_c)^2+R^2)((r_c+r_a)^2+R^2)>16\sqrt{3}\cdot F^3$$

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Solution by Titu Zvonaru-Romania

It is known the inequality of Arkady Alt (see for example [1]):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x=y=z, t=x\sqrt{2}$.

Since $r_a=\frac{F}{s-a}$, $F=sr$ and $ab+bc+ca=s^2+r^2+4Rr$ obtain

$$r_a+r_b+r_c=\frac{F}{s-a}+\frac{F}{s-b}+\frac{F}{s-c}=$$

$$=\frac{F(s^2-s(b+c)+bc+s^2-s(c+a)+ca+s^2-s(a+b)+ab)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs(3s^2-4s^2+s^2+r^2+4Rr)}{s(s-a)(s-b)(s-c)}=\frac{Fsr(4R+r)}{F^2}=4R+r \quad (2)$$

Using (1), (2), the inequality $r(4R+r)\ge F\sqrt{3}$ (item 7.2 from [2]) and Euler inequality $R\ge 2r$, it follows that

$$((r_a+r_b)^2+R^2)((r_b+r_c)^2+R^2)((r_c+r_a)^2+R^2)>$$

$$>\frac{3}{4}R^4(r_a+r_b+r_b+r_c+r_c+r_a)^2=3R^4(4R+r)^2\ge$$

$$\ge 12R^2\big(r(4R+r)\big)^2\ge 36R^2F^2\ge 16r(4R+r)F^2\ge 16\sqrt{3}\cdot F^3.$$

Since in an equilateral triangle we have $R\ne 2r_a\sqrt{2}$, inequality is strict.

[1] D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024

[2] O. Bottema, *Geometric Inequalities*, Groningen 1969

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.