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J.2396 If I − incenter then in ΔABC holds:

$$2\sqrt{3}r \le \sqrt{a\cdot AI^2 + b\cdot BI^2 + c\cdot CI^2} \le R\sqrt{3}$$

Proposed by George Apostolopoulos – Greece

Solution by Titu Zvonaru-Romania

Since $AI = \frac{bc\cos\frac{A}{2}}{s}$, we obtain:

$$a\cdot AI^2 + b\cdot BI^2 + c\cdot CI^2 = \frac{abc(s-a)}{s} + \frac{abc(s-b)}{s} + \frac{abc(s-c)}{s} = abc.$$

We have $24\sqrt{3}r^3 \le abc \Leftrightarrow 6\sqrt{3}r^2 \le Rs$,

which is true because $3\sqrt{3}r \le s$ (item 5.11 from $[1]$) and $2r \le R$.

The right inequality is equivalent to $abc \le 3\sqrt{3}R^3 \Leftrightarrow 4rs \le 3\sqrt{3}R^2$,

which is true because $2s \le 3\sqrt{3}R$ (item 5.3 from $[1]$) and $2r \le R$.

Equality holds if and only if ΔABC is equilateral.

[1] O. Bottema, *Geometric Inequalities*, Groningen 1969