ROMANIAN MATHEMATICAL MAGAZINE

J.2399 In acute ΔABC holds:

$$\frac{b^2+c^2}{a^2}\cos A+\frac{c^2+a^2}{b^2}\cos B+\frac{a^2+b^2}{c^2}\cos A\ge 24\left(\frac{r}{R}\right)^2-3$$

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Since by $AM-GM$ inequality we have $b^2+c^2\ge 2bc$ and $2bc\cos A=b^2+c^2-a^2$, it follows that

$$\frac{b^2+c^2}{a^2}\cos A+\frac{c^2+a^2}{b^2}\cos B+\frac{a^2+b^2}{c^2}\cos A\ge$$

$$\ge\frac{2bc\cos A}{a^2}+\frac{2ca\cos B}{b^2}+\frac{2ab\cos C}{c^2}=$$

$$=\frac{b^2+c^2-a^2}{a^2}+\frac{c^2+a^2-b^2}{b^2}+\frac{a^2+b^2-c^2}{c^2}=$$

$$=\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}+\frac{c^2}{a^2}+\frac{a^2}{c^2}-3\ge 2+2+2-3=3.$$

It remains to prove that:

$$3\ge 24\left(\frac{r}{R}\right)^2-3\Leftrightarrow 6\ge 24\left(\frac{r}{R}\right)^2\Leftrightarrow R^2\ge 4r^2\Leftrightarrow R\ge 2r,$$

that is Euler's inequality.

Equality holds if and only if $a=b=c$.