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J.2401 If $a \in R_+, m \geq 1$ and $b, c, d, x, y, z \in R_+^*, X = x + y + z,$

$cX > d\max\{x, y, z\}$, then:

$$\frac{aX + bx}{(cX - dx)^m} + \frac{aX + by}{(cX - dy)^m} + \frac{aX + bz}{(cX - dz)^m} \geq \frac{3^m(3a + b)}{(3c - d)^m X^{m-1}}.$$

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Applying Radon's inequality we obtain:

$$\frac{aX}{(cX - dx)^m} + \frac{aX}{(cX - dy)^m} + \frac{aX}{(cX - dz)^m} =$$

$$= aX\left(\frac{1^{m+1}}{(cX - dx)^m} + \frac{1^{m+1}}{(cX - dy)^m} + \frac{1^{m+1}}{(cX - dz)^m}\right)$$

$$\geq aX \cdot \frac{3^{m+1}}{\big(3cX - d(x + y + z)\big)^m} = aX \cdot \frac{3^{m+1}}{(3cX - dX)^m} = \frac{3^{m+1}a}{(3c - d)^m X^{m-1}} \quad (1)$$

Using again Radon's inequality and the known inequality:

$3(x^2 + y^2 + z^2) \geq (x + y + z)^2 = X^2$, yields that

$$\frac{bx}{(cX - dx)^m} + \frac{by}{(cX - dy)^m} + \frac{bz}{(cX - dz)^m} =$$

$$= \frac{bx^{m+1}}{(cxX - dx^2)^m} + \frac{by^{m+1}}{(cyX - dy^2)^m} + \frac{bz^{m+1}}{(czX - dz^2)^m} \geq$$

$$\geq \frac{b(x + y + z)^{m+1}}{\big(c(x + y + z)X - d(x^2 + y^2 + z^2)\big)^m} = \frac{b(x + y + z)^{m+1}}{\big(cX^2 - d(x^2 + y^2 + z^2)\big)^m} \geq$$

$$\geq \frac{b(x + y + z)^{m+1}}{\left(cX^2 - d \cdot \frac{X^2}{3}\right)^m} = \frac{b \cdot X^{m+1} \cdot 3^m}{(3c - d)^m X^{2m}} = \frac{b \cdot 3^m}{(3c - d)^m X^{m-1}} \quad (2)$$

Adding the inequalities (1) and (2) it follows that

$$\frac{aX + bx}{(cX - dx)^m} + \frac{aX + by}{(cX - dy)^m} + \frac{aX + bz}{(cX - dz)^m} \geq \frac{3^{m+1}a}{(3c - d)X^{m-1}} + \frac{b \cdot 3^m}{(3c - d)^m X^{m-1}}$$
$$= \frac{3^m(3a + b)}{(3c - d)^m X^{m-1}}.$$

Equality holds if and only if $x = y = z, m = 1$.