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J.2403 If $m \geq 0, x, y, z, t > 0$, then in any ΔABC, holds

$$\sum_{\text{cyc}} \frac{\left(xm_a^2 + ym_b^2\right)^{m+1}}{\left(zh_a^2 + th_b^2\right)^m} \geq \frac{x+y}{z+t} 3\sqrt{3}F$$

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We have $h_a \leq m_a$ and $m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2+b^2+c^2)}{4}$.

Applying Radon's inequality and Ionescu – Weitzenbock's inequality, we obtain:

$$\sum_{\text{cyc}} \frac{\left(xm_a^2 + ym_b^2\right)^{m+1}}{\left(zh_a^2 + th_b^2\right)^m} \overset{RADON}{\underset{(1)}{\geq}} \frac{\left((x+y)\left(m_a^2 + m_b^2 + m_c^2\right)\right)^{m+1}}{\left((z+t)\left(m_a^2 + m_b^2 + m_c^2\right)\right)^m} =$$

$$= \frac{3(x+y)(a^2+b^2+c^2)}{4(z+t)} \overset{IONESCU\ WEITZENBOCK}{\underset{(2)}{\geq}} \frac{3(x+y)\left(4\sqrt{3}F\right)}{4(z+t)} = \frac{x+y}{z+t} 3\sqrt{3}F.$$

Equality holds if and only if $h_a = m_a, h_b = m_b, h_c = m_c$,

hence if and only if ΔABC is equilateral.