ROMANIAN MATHEMATICAL MAGAZINE

J.2410 For $x \in [-1,1]$, prove that

$$\left|\frac{\left(x+\sqrt{x^2-1}\right)^{n+1}+\left(x-\sqrt{x^2-1}\right)^{n+1}+\left(x+\sqrt{x^2-1}\right)^{n-1}+\left(x-\sqrt{x^2-1}\right)^{n-1}}{\left(x+\sqrt{x^2-1}\right)^{n}+\left(x-\sqrt{x^2-1}\right)^{n}}\right|\le 2$$

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Since $x \in [-1,1]$, there exists $u \in [0°, 180°]$ such that $x = \cos u$. We obtain

$$x+\sqrt{x^2-1}=\cos u+i\sin u,\ x-\sqrt{x^2-1}=\cos u-i\sin u=\cos(-u)+i\sin(-u).$$

By Moivre's formula, it follows that

$$\left(x+\sqrt{x^2-1}\right)^{n}=(\cos u+i\sin u)^n=\cos(nu)+i\sin(nu),$$

$$\left(x-\sqrt{x^2-1}\right)^{n}=\cos(-nu)+i\sin(-nu)=\cos(nu)-i\sin(nu).$$

It follows that

$$\left|\frac{\left(x+\sqrt{x^2-1}\right)^{n+1}+\left(x-\sqrt{x^2-1}\right)^{n+1}+\left(x+\sqrt{x^2-1}\right)^{n-1}+\left(x-\sqrt{x^2-1}\right)^{n-1}}{\left(x+\sqrt{x^2-1}\right)^{n}+\left(x-\sqrt{x^2-1}\right)^{n}}\right|$$

$$=\left|\frac{2\cos(n+1)u+2\cos(n-1)u}{2\cos nu}\right|=\left|\frac{2\cos\frac{(n+1)u+(n-1)u}{2}\cos\frac{(n+1)u-(n-1)u}{2}}{\cos nu}\right|=$$

$$=|2\cos u|\le 2.$$

Equality holds if and only if $\cos u = 1$, that is $x = 0$.