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J.2414 If $m \geq 0$ and $x, y > 0$ then in triangle ABC holds:

$$(a^{m+1}+b^{m+1}+c^{m+1})\left(\frac{1}{(ax+y\sqrt{bc})^m}+\frac{1}{(bx+y\sqrt{ca})^m}+\frac{1}{(cx+y\sqrt{ab})^m}\right) \geq \frac{6(3)^{\frac{3}{4}}}{(x+y)^m}\cdot\sqrt{F}.$$

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By Power Mean inequality we get:

$$\left(\frac{a^{m+1}+b^{m+1}+c^{m+1}}{3}\right)^{\frac{1}{m+1}} \geq \frac{a+b+c}{3} \Leftrightarrow$$

$$\Leftrightarrow 3^m(a^{m+1}+b^{m+1}+c^{m+1}) \geq (a+b+c)^{m+1} \quad (1)$$

Applying Radon's inequality and the well known inequality

$xy+yz++zx \leq x^2+y^2+z^2$ (for $\sqrt{a},\sqrt{b},\sqrt{c}$) we obtain:

$$\frac{1}{(ax+y\sqrt{bc})^m}+\frac{1}{(bx+y\sqrt{ca})^m}+\frac{1}{(cx+y\sqrt{ab})^m}=$$

$$=\frac{1^{m+1}}{(ax+y\sqrt{bc})^m}+\frac{1^{m+1}}{(bx+y\sqrt{ca})^m}+\frac{1^{m+1}}{(cx+y\sqrt{ab})^m}\geq$$

$$\geq \frac{(1+1+1)^{m+1}}{\left(x(a+b+c)+y(\sqrt{bc}+\sqrt{ca}+\sqrt{bc})\right)^m}\geq$$

$$\geq \frac{3^{m+1}}{\left(x(a+b+c)+y(a+b+c)\right)^m}=\frac{3^{m+1}}{(x+y)^m(a+b+c)^m} \quad (2)$$

Using $(1),(2)$ and Hadwiger inequality $s^2 \geq 3\sqrt{3}F$ (item $4.2$from $[1]$) it follows that

$$(a^{m+1}+b^{m+1}+c^{m+1})\left(\frac{1}{(ax+y\sqrt{bc})^m}+\frac{1}{(bx+y\sqrt{ca})^m}+\frac{1}{(cx+y\sqrt{ab})^m}\right)\geq$$

$$\geq \frac{(a+b+c)^{m+1}}{3^m}\cdot\frac{3^{m+1}}{(x+y)^m(a+b+c)^m}=\frac{3(a+b+c)}{(x+y)^m}\geq\frac{6(3)^{3/4}}{(x+y)^m}\cdot\sqrt{F}.$$

Equality holds if and only if the triangle ABC is equilateral.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969