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J.2416 If $m \geq 0$ and in triangle $ABC, A_1 \in BC, B_1 \in CA, C_1 \in AB$ such that $BA_1 = mA_1C, CB_1 = mB_1A, AC_1 = mC_1B$, then:

$$\frac{a \cdot A_1B_1^2}{h_b} + \frac{b \cdot BC_1^2}{h_c} + \frac{c \cdot C_1A_1^2}{h_a} \geq \frac{8m}{(m+1)^2} \cdot F$$

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Since $BA_1 = mA_1C, BA_1 + A_1C = a$, yields that $BA_1 = \frac{am}{m+1}, A_1C = \frac{a}{m+1}$. We obtain:

$$[A_1B_1C_1] = [ABC] - [AB_1C_1] - [BC_1A_1] - [CA_1B_1] =$$

$$= F - \frac{AB_1 \cdot AC_1 \sin A}{2} - \frac{BC_1 \cdot BA_1 \sin B}{2} - \frac{CA_1 \cdot CB_1 \sin C}{2} =$$

$$= F - \frac{bcm\sin A}{2(m+1)^2} - \frac{cam\sin B}{2(m+1)^2} - \frac{abm\sin C}{2(m+1)^2} = F - \frac{mF}{(m+1)^2} - \frac{mF}{(m+1)^2} - \frac{mF}{(m+1)^2} =$$

$$= F - \frac{3mF}{(m+1)^2} = \frac{m^2 - m + 1}{(m+1)^2} \cdot F \quad (1)$$

We have $ah_a = bh_b = ch_c = 2F$.

Applying (1), $AM - GM$ inequality and Carlitz's inequality $(abc)^{2/3} \geq \frac{4}{\sqrt{3}}F$ (for the triangles ABC and $A_1B_1C_1$), it follows that:

$$\frac{a \cdot A_1B_1^2}{h_b} + \frac{b \cdot BC_1^2}{h_c} + \frac{c \cdot C_1A_1^2}{h_a} = \frac{ab \cdot A_1B_1^2}{bh_b} + \frac{bc \cdot BC_1^2}{ch_c} + \frac{ca \cdot C_1A_1^2}{ah_a} =$$

$$= \frac{ab \cdot A_1B_1^2}{2F} + \frac{bc \cdot BC_1^2}{2F} + \frac{ca \cdot C_1A_1^2}{2F} \geq \frac{1}{2F} \cdot 3(abc)^{\frac{2}{3}} \cdot (A_1B_1 \cdot B_1C_1 \cdot C_1A_1)^{\frac{2}{3}} \geq$$

$$\geq \frac{3}{2F}\left(\frac{4}{\sqrt{3}}F\right)\left(\frac{4}{\sqrt{3}} \cdot \frac{m^2 - m + 1}{(m+1)^2} \cdot F\right) = \frac{8(m^2 - m + 1)}{(m+1)^2} \cdot F \geq \frac{8m}{(m+1)^2} \cdot F.$$

Equality holds if and only the triangle ABC equilateral and $m = 1$, that is if and only if $A_1B_1C_1$ is median triangle of equilateral triangle ABC.