ROMANIAN MATHEMATICAL MAGAZINE

J.2417 If $x \geq 0$ and in $\Delta ABC, A_1 \in BC, B_1 \in CA, C_1 \in AB$ such that

$$BA_1 = xA_1C, CB_1 = xB_1A, AC_1 = xC_1B, \text{ then:}$$

$$[A_1B_1C_1] = \frac{x^2 - x + 1}{(x+1)^2} \cdot F \geq \frac{x}{(x+1)^2} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

Solution by Titu Zvonaru-Romania

By $BA_1 = xA_1C, BA_1 + A_1C = a$, we get:

$BA_1 = \frac{ax}{x+1}, A_1C = \frac{a}{x+1}$, and similar $CB_1 = \frac{bx}{x+1}, B_1A = \frac{b}{x+1}, AC_1 = \frac{cx}{x+1}, C_1B = \frac{c}{x+1}$.

We obtain $[AB_1C_1] = \frac{AB_1AC_1 \sin A}{2} = \frac{bcx \sin A}{2(x+1)^2} = \frac{x}{(x+1)^2} \cdot F$. It follows that

$$[A_1B_1C_1] = [ABC] - [AB_1C_1] - [BC_1A_1] - [CA_1B_1] = F - \frac{3x}{(x+1)^2} \cdot F$$
$$= \frac{x^2 - x + 1}{(x+1)^2} \cdot F.$$

By $AM - GM$ we have $x^2 - x + 1 = x^2 + 1 - x \geq 2x - x = x$. It results that

$$[A_1B_1C_1] = \frac{x^2 - x + 1}{(x+1)^2} \cdot F \geq \frac{x}{(x+1)^2} \cdot F.$$

Equality holds if and only if $x = 1$, that is if and only if $\Delta A_1B_1C_1$ is the median triangle of ΔABC.