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J.2419 If $x,y,z>0$ then in triangle ABC

$$\frac{w_a^x\cdot a^{x+y}\cdot b^z}{h_b^z}+\frac{w_b^x\cdot b^{x+y}\cdot c^z}{h_c^z}+\frac{w_c^x\cdot c^{x+y}\cdot a^z}{h_a^z}\ge 2^{x+2z}\cdot F^{\frac{2x+y}{2}}\cdot 3^{\frac{4-y-2z}{4}}$$

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We have $ah_a=bh_b=ch_c=2F, w_a\ge h_a, w_b\ge h_b, w_c\ge h_c$.

Applying $AM-GM$ inequality and Carlitz's inequality $(abc)^{2/3}\ge\frac{4}{\sqrt{3}}F$, it follows that:

$$\frac{w_a^x\cdot a^{x+y}\cdot b^z}{h_b^z}+\frac{w_b^x\cdot b^{x+y}\cdot c^z}{h_c^z}+\frac{w_c^x\cdot c^{x+y}\cdot a^z}{h_a^z}\ge\frac{h_a^x\cdot a^{x+y}\cdot b^z}{h_b^z}+\frac{h_b^x\cdot b^{x+y}\cdot c^z}{h_c^z}+\frac{h_c^x\cdot c^{x+y}\cdot a^z}{h_a^z}=$$

$$=\frac{(ah_a)^x\cdot a^y\cdot b^{2z}}{(bh_b)^z}+\frac{(bh_b)^x\cdot b^y\cdot c^{2z}}{(ch_c)^z}+\frac{(ch_c)^x\cdot c^y\cdot a^{2z}}{(ah_a)^z}=$$

$$=\frac{(2F)^x}{(2F)^z}\left(a^y\cdot b^{2z}+b^y\cdot c^{2z}+c^y\cdot a^{2z}\right)\ge(2F)^{x-z}\cdot 3(abc)^{\frac{y+2z}{3}}\ge$$

$$\ge(2F)^{x-z}\cdot 3\left(\frac{4}{\sqrt{3}}F\right)^{\frac{y+2z}{2}}=2^{x+2z}\cdot F^{\frac{2x+y}{2}}\cdot 3^{\frac{4-y-2z}{4}}.$$

Equality holds if and only if triangle ABC is equilateral and $x=y=z$.