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J.2420 If $a, b, c > 0, a + b + c = 3$, then

$$\sum_{cyc} \frac{1}{a^3 + b^3} + 3 \sum_{cyclic} \frac{1}{ab(a+b)} \geq 6.$$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

Solution by Titu Zvonaru-Romania

Applying Bergström inequality we have:

$$\sum_{cyc} \frac{1}{a^3 + b^3} + 3 \sum_{cyclic} \frac{1}{ab(a+b)} = \sum_{cyc} \left(\frac{1}{a^3 + b^3} + \frac{1}{ab(a+b)} + \frac{1}{ab(a+b)} + \frac{1}{ab(a+b)} \right)$$

$$\geq \sum_{cyclic} \frac{(1+1+1+1)^2}{a^3 + b^3 + ab(a+b) + ab(a+b) + ab(a+b)} = \sum_{cyc} \frac{16}{(a+b)^3}.$$

It suffices to prove that: $\frac{1}{(a+b)^3} + \frac{1}{(b+c)^3} + \frac{1}{(c+a)^3} \geq \frac{3}{8}$ $\quad (1)$

Here two proofs for (1).

I. By $AM - GM$ inequality it follows that:

$$\frac{1}{(a+b)^3} + \frac{1}{(b+c)^3} + \frac{1}{(c+a)^3} \geq \frac{3}{(a+b)(b+c)(c+a)} \geq \frac{3}{\left(\frac{a+b+b+c+c+a}{3}\right)^3} = \frac{3}{\left(\frac{6}{3}\right)^3} = \frac{3}{8}.$$

II. Using line tangent method, we deduce that for $x > 0$ the following inequality holds:

$$\frac{1}{(3-x)^3} \geq \frac{3x-1}{16} \quad (2)$$

Indeed, (2) is equivalent to $3x^4 - 28x^3 + 90x^2 - 108x + 43 \geq 0 \Leftrightarrow$

$$(x-1)^2(3x^2 - 22x + 43) \geq 0.$$

The last inequality is true because the discriminant of the quadratic

$f(x) = 3x^2 - 22x + 43$ is equal to $22^2 - 4 \cdot 3 \cdot 43 = 4(11^2 - 3 \cdot 43) = -4 \cdot 8 < 0$.

By (2) it results that

$$\frac{1}{(a+b)^3} + \frac{1}{(b+c)^3} + \frac{1}{(c+a)^3} = \frac{1}{(3-c)^3} + \frac{1}{(3-a)^3} + \frac{1}{(3-b)^3} \geq \frac{3c-1}{16} + \frac{3a-1}{16} + \frac{3b-1}{16}$$
$$= \frac{3(a+b+c)-3}{16} = \frac{3}{8}.$$

Equality holds if and only if $a = b = c = 1$.