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J.2426 If $a,b,c\geq 0, a+b+c=2$ then:

$$a^2b^2+b^2c^2+c^2a^2\leq 1.$$

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Since $a+b+c=2$, the inequality is equivalent to

$$16(a^2b^2+b^2c^2+c^2a^2)\leq (a+b+c)^4 \quad (1)$$

Here are two proofs for (1):

I. Since the inequality is symmetric and homogeneous of 4 degree, it suffices to prove for

$c=0$ and $b=c$.

For $c=0$ we have to prove that $16a^2b^2\leq (a+b)^4$, true by $AM-GM$ inequality.

For $b=c$ we have to prove that

$$16(2a^2b^2+b^4)\leq (a+2b)^4$$

$$a(a^3+8a^2b-8ab^2+32b^3)\geq 0 \Leftrightarrow a((a-b)^2(a+10b)+11ab^2+22b^3)\geq 0$$

II. The inequality (1) is equivalent to

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)+5ab(a-b)^2+5bc(b-c)^2+5ca(c-a)^2+11abc(a+b+c)\geq 0,$$

which is true by Schur inequality

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0.$$

Equality holds if and only if $(a,b,c)=(0,1,1),(1,0,1),(1,1,0)$.