ROMANIAN MATHEMATICAL MAGAZINE

J.2427 If $x, y, z > 0, x + y + z = 3$, then:

$$2\sum_{cyc} x\left(y + \frac{1}{y} + \frac{1}{y^3}\right) \geq 3\sum_{cyc}\frac{x}{y^2} + 9.$$

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Applying $AM - GM$ inequality and Radon's inequality, it follows that:

$$2\sum_{cyc} x\left(y + \frac{1}{y} + \frac{1}{y^3}\right) \geq 2\sum_{cyc}\left(xy + \frac{2x}{y^2}\right) = 2\sum_{cyc} xy + 3\sum_{cyc}\frac{x}{y^2} + \sum_{cyc}\frac{x}{y^2} =$$

$$= 2\sum_{cyc} xy + 3\sum_{cyc}\frac{x}{y^2} + \sum_{cyc}\frac{x^3}{(xy)^2} \geq 2\sum_{cyc} xy + 3\sum_{cyc}\frac{x}{y^2} + \frac{(x+y+z)^3}{(xy+yz+zx)^2} =$$

$$= 3\sum_{cyc}\frac{x}{y^2} + (xy+yz+zx) + (xy+yz+zx) + \frac{27}{(xy+yz+zx)^2} \geq$$

$$\geq 3\sum_{cyc}\frac{x}{y^2} + 3\left((xy+yz+zx)\cdot(xy+yz+zx)\cdot\frac{27}{(xy+yz+zx)^2}\right)^{\frac{1}{3}} \geq 3\sum_{cyc}\frac{x}{y^2} + 9.$$

Equality holds if and only if $x = y = z = 1$.