ROMANIAN MATHEMATICAL MAGAZINE

**J.2428. If $a,b,c\geq 0, a+b+c=2$ then:**

$$a^2b^2+b^2c^2+c^2a^2+\frac{11}{8}abc\leq 1.$$

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Since $a+b+c=2$, the inequality is equivalent to

$$16\left(a^2b^2+b^2c^2+c^2a^2\right)+11abc(a+b+c)\leq (a+b+c)^4 \quad (1)$$

Here two proofs for $(1)$:

I. Since the inequality is symmetric and homogeneous of 4 degree, it suffices to prove for $c=0$ and $b=c$.

For $c=0$ we have to prove that $16a^2b^2\leq (a+b)^4$, true by $AM-GM$ inequality.

For $b=c$ we have to prove that

$$16\left(2a^2b^2+b^4\right)+11ab^2(a+2b)\leq (a+2b)^4$$

$$a\left(a^3+8a^2b-19ab^2+10b^3\right)\geq 0 \Leftrightarrow a(a-b)^2(a+10b)\geq 0$$

II. The inequality $(1)$ is equivalent to

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)+5ab(a-b)^2 + 5bc(b-c)^2+5ca(c-a)^2\geq 0,$$

which is true by Schur inequality

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0.$$

Equality holds if and only if $(a,b,c)=\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right),(0,1,1),(1,0,1),(1,1,0)$.