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J.2434 If $a,b,c>0,\ abc=1$ then

$$(a^3+1)(b^3+1)(c^3+1)\geq\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)$$

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Solution by Titu Zvonaru-Romania

Since $abc=1$, there are positive real numbers x,y,z such that $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$.

The given inequality is echivalent to

$$(x^3+y^3)(y^3+z^3)(z^3+x^3)\geq xyz(x^2+y^2)(y^2+z^2)(z^2+x^2)$$

$$\sum_{\text{sym}}x^6y^3\geq\sum_{\text{sym}}x^5y^3z,$$

which is true by Muirhead's inequality $(6,3,0)\geq(5,3,1)$.

Equality holds if and only if $x=y=z$, that is if and only if $a=b=c=1$.