ROMANIAN MATHEMATICAL MAGAZINE

J.2445 If $x, y, z > 0, x^2 + y^2 + z^2 \leq 3$, then

$$\frac{1}{7-x} + \frac{1}{7-y} + \frac{1}{7-z} \leq \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

Solution by Titu Zvonaru-Romania

We will prove a refinement :

Prove that if $x, y, z \leq 5$, then $\frac{1}{7-x} + \frac{1}{7-y} + \frac{1}{7-z} \leq \frac{x^2+y^2+z^2+33}{72}$.

By tangent line method we have $\frac{1}{7-x} \leq \frac{x^2+11}{72}$ $\quad (1)$

For $x, y, z \leq 5$, the inequality (1) is equivalent to $(x-1)^2(x-5) \leq 0$. It follows that

$$\frac{1}{7-x} + \frac{1}{7-y} + \frac{1}{7-z} \leq \frac{x^2+11}{72} + \frac{y^2+11}{72} + \frac{z^2+11}{72} = \frac{x^2+y^2+z^2+33}{72} \leq \frac{36}{72} = \frac{1}{2}$$

Equality holds if and only if

$$(x, y, z) = (1,1,1), (1,1,5), (1,5,1), (5,1,1), (1,5,5), (5,1,5), (5,5,1), (5,5,5)$$