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J.2446 If $m, n \in R_+^*$, then in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{m+n\cot^2\frac{A}{2}}+\frac{\tan\frac{B}{2}}{m+n\cot^2\frac{B}{2}}+\frac{\tan\frac{C}{2}}{m+n\cot^2\frac{C}{2}}\geq\frac{(4R+r)^2r}{m(4R+r)r+ns^2}$$

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Using formulas:

$$\tan\frac{A}{2}=\frac{r}{s-a},\cot\frac{A}{2}=\frac{s-a}{r},ab+bc+ca=s^2+r^2+4Rr,$$

$$F^2=s(s-a)(s-b)(s-c)$$

we obtain:

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r}{s-a}+\frac{r}{s-b}+\frac{r}{s-c}=$$

$$=\frac{r\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{r(3s^2-s(a+b+b+c+c+a)+ab+bc+ca)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{sr(3s^2-4s^2+s^2+r^2+4Rr)}{s(s-a)(s-b)(s-c)}=\frac{Fr(4R+r)}{F^2}=\frac{r(4R+r)}{sr}=\frac{4R+r}{s}$$

and

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{s}{r}.$$

Applying Bergström inequality it follows that

$$\frac{\tan\frac{A}{2}}{m+n\cot^2\frac{A}{2}}+\frac{\tan\frac{B}{2}}{m+n\cot^2\frac{B}{2}}+\frac{\tan\frac{C}{2}}{m+n\cot^2\frac{C}{2}}=$$

$$=\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\tan\frac{A}{2}\cot^2\frac{A}{2}}+\frac{\tan^2\frac{B}{2}}{m\tan\frac{B}{2}+n\tan\frac{B}{2}\cot^2\frac{B}{2}}+\frac{\tan^2\frac{C}{2}}{m\tan\frac{C}{2}+n\tan\frac{C}{2}\cot^2\frac{C}{2}}=$$

$$=\frac{\tan^2\frac{A}{2}}{m\tan\frac{A}{2}+n\cot\frac{A}{2}}+\frac{\tan^2\frac{B}{2}}{m\tan\frac{B}{2}+n\cot\frac{B}{2}}+\frac{\tan^2\frac{C}{2}}{m\tan\frac{C}{2}+n\cot\frac{C}{2}}=$$

$$\geq\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}{m\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)+n\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)}=$$

$$=\frac{\frac{(4R+r)^2}{s^2}}{\frac{m(4R+r)}{s}+\frac{ns}{r}}=\frac{(4R+r)^2r}{m(4R+r)r+ns^2}.$$

Equality holds if and only if ΔABC is equilateral.