ROMANIAN MATHEMATICAL MAGAZINE

J.2447 If $x,y,z\in R_+^*$, then in any triangle ABC holds:

$$\frac{\cot^3\frac{A}{2}}{x\tan\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{x\tan\frac{B}{2}+y\tan\frac{C}{2}+z\tan\frac{C}{2}\tan\frac{A}{2}}+$$

$$+\frac{\cot^3\frac{C}{2}}{x\tan\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{s^4}{((4R+r)^2x+(y-2x)s^2+3zrs)r^2}$$

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Solution by Titu Zvonaru-Romania

Using formulas

$$\tan\frac{A}{2}=\frac{r}{s-a},\cot\frac{A}{2}=\frac{s-a}{r},ab+bc+ca=s^2+r^2+4Rr,F^2=s(s-a)(s-b)(s-c)$$

we obtain

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r}{s-a}+\frac{r}{s-b}+\frac{r}{s-c}=$$

$$=\frac{r\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{r\big(3s^2-s(a+b+b+c+c+a)+ab+bc+ca\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{sr(3s^2-4s^2+s^2+r^2+4Rr)}{s(s-a)(s-b)(s-c)}=\frac{Fr(4R+r)}{F^2}=\frac{r(4R+r)}{sr}=\frac{4R+r}{s},$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=\frac{r^2}{(s-a)(s-b)}+\frac{r^2}{(s-b)(s-c)}+\frac{r^2}{(s-c)(s-a)}=1,$$

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r^3}{(s-a)(s-b)(s-c)}=\frac{r^3s}{s(s-a)(s-b)(s-c)}=\frac{r}{s}$$

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{3s-a-b-c}{r}=\frac{s}{r}.$$

Applying Bergström inequality it follows that

$$\frac{\cot^3\frac{A}{2}}{x\tan\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{x\tan\frac{B}{2}+y\tan\frac{C}{2}+z\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\cot^3\frac{C}{2}}{x\tan\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}}=$$

$$=\frac{\tan\frac{A}{2}\cot^3\frac{A}{2}}{x\tan^2\frac{A}{2}+y\tan\frac{A}{2}\tan\frac{B}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}\cot^3\frac{B}{2}}{x\tan^2\frac{B}{2}+y\tan\frac{B}{2}\tan\frac{C}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}\cot^3\frac{C}{2}}{x\tan^2\frac{C}{2}+y\tan\frac{C}{2}\tan\frac{A}{2}+z\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}}\geq$$

$$\geq\frac{\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2}{x\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)+y\left(\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right)+3z\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$$

$$=\frac{\frac{s^2}{r^2}}{x\left(\frac{4R+r}{s}\right)^2-2x+y+\frac{3zr}{s}}=\frac{s^4}{((4R+r)^2x+(y-2x)s^2+3zrs)r^2}.$$

Equality holds if and only if ΔABC is equilateral.